On the Complexity of Equilibrium Computation in First-Price Auctions

From the following two works:

F., Giannakopoulos, Hollender, Lazos, and Poças. *On the Complexity of Equilibrium Computation in First-Price Auctions*. EC 2021, SICOMP 2023.
F., Giannakopoulos, Hollender, and Kokkalis. *On the Computation of Equilibria in Discrete First-Price Auctions*. EC 2024.

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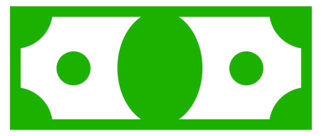 = £200,000

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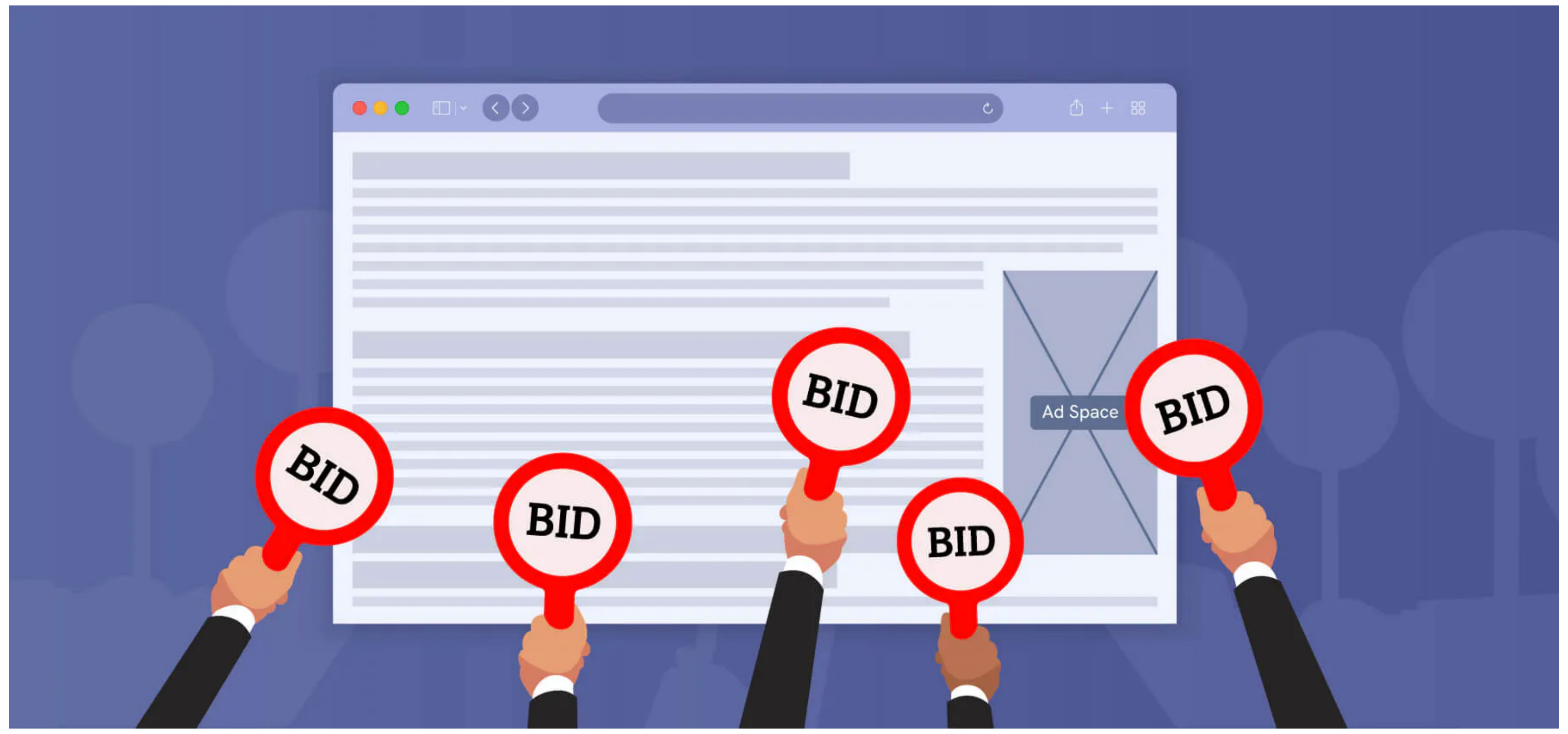
- The winner pays her bid, the other bidders pay zero.

First Price Auctions in practice



Google's First-Price Auction Switch Is Making Header Bidding Partners Win More

by Sarah Sluis // Thursday, September 5th, 2019 – 6:00 am

ARTICLES

Everything you should know about Google's move to First Price

NEWS

Google outlines move to first-price auction for Ad Manager

Who else is interested?

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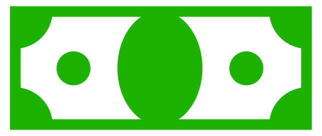 = £200,000

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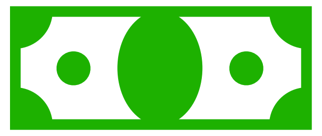 = £200,000

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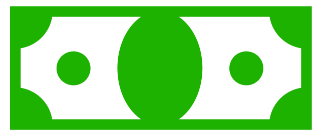
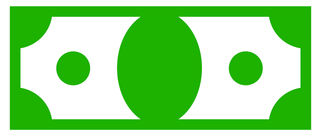
 = £200,000

 = £150,000

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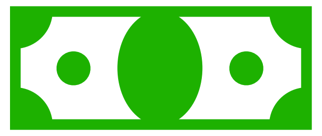
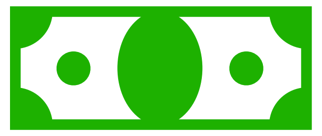
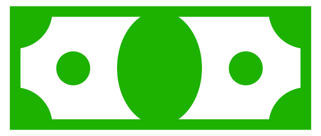
 ~~= £2[illegible],000~~

 = £151,000

 = £150,000

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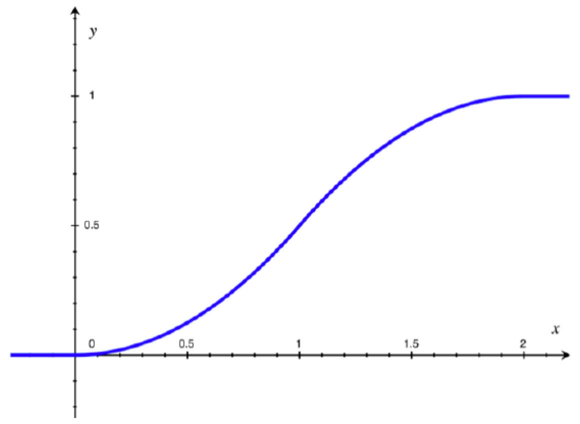
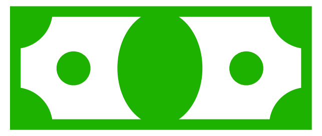


How much should I bid?

I have incentives to underbid, but I also don’t want to lose the house, or at least not with high probability.



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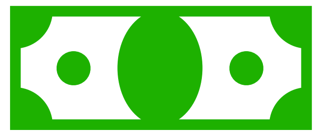

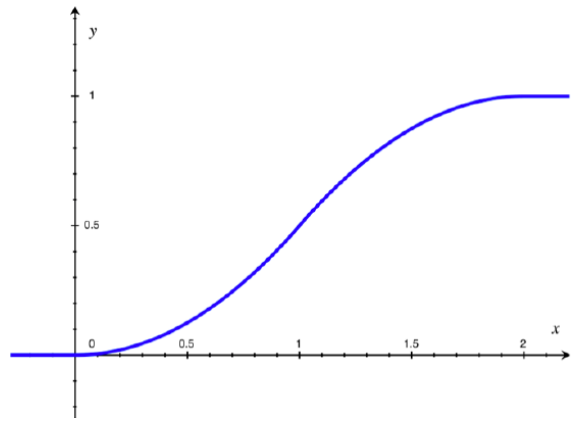

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- Pure strategy: $\beta_i : V \to B$

- Ex-post utility: $\tilde{u}_i(\mathbf{b}; v_i) := \begin{cases} \frac{1}{|W(\mathbf{b})|}(v_i - b_i), & \text{if } i \in W(\mathbf{b}), \\ 0, & \text{otherwise,} \end{cases}$ where $W(\mathbf{b}) = \text{argmax}_{j \in N} b_j$

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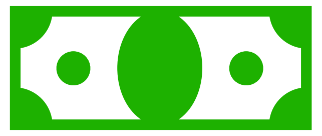

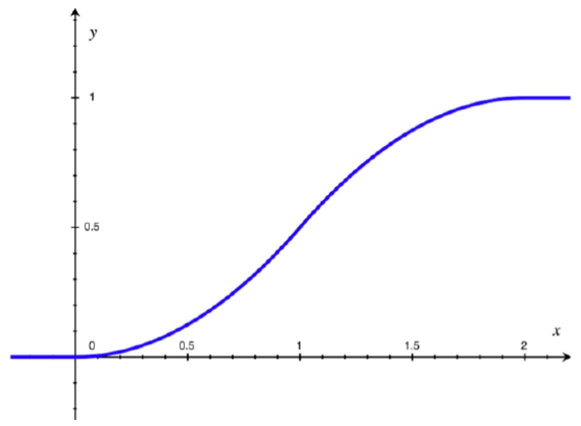

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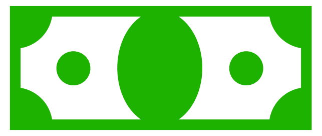

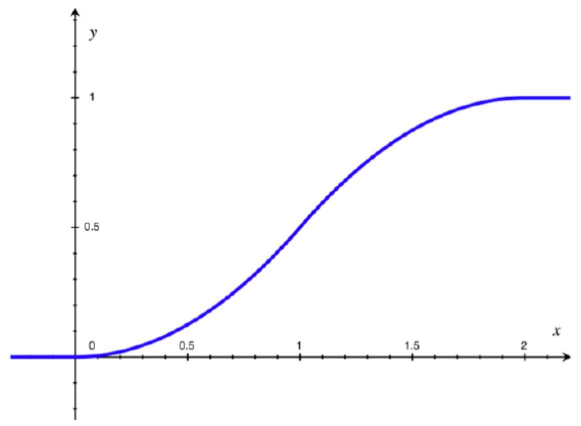

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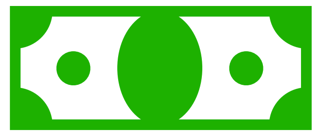

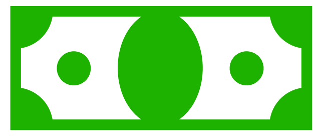

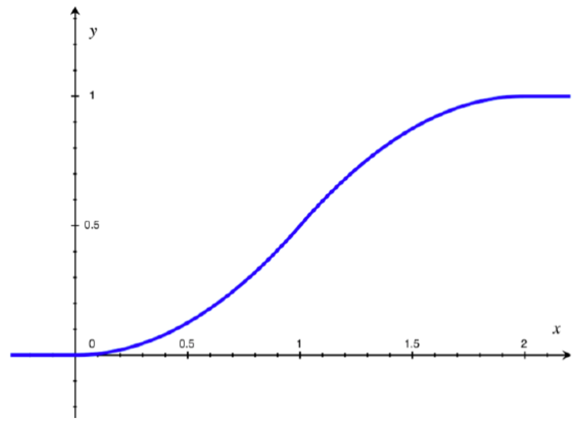

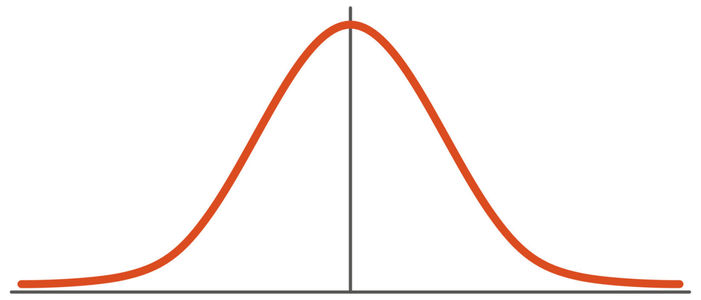

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This is called a Bayes-Nash Equilibrium.

Bayes-Nash Equilibrium

- A strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ is an ε-approximate pure Bayes-Nash Equilibrium if for any bidder $i \in N$, any value $v_i \in V$, and any bid $b \in B$:

$$u_i(\beta_i(v_i), \beta_{-\mathbf{i}}; v_i) \geq u_i(b, \beta_{-\mathbf{i}}; v_i) - \varepsilon$$

We refer to a 0-approximate PBNE as an *exact* PBNE

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  - e.g., at least as hard as finding mixed Nash equilibria in normal form games, market equilibria in exchange markets, etc.

Complexity Landscape

continuous priors

iid

IPV

subjective priors

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continuous priors

PBNE: PPAD-complete [FGHLP23]

iid

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Complexity Landscape

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PBNE (trilateral tie-breaking): PPAD-complete [CP23]

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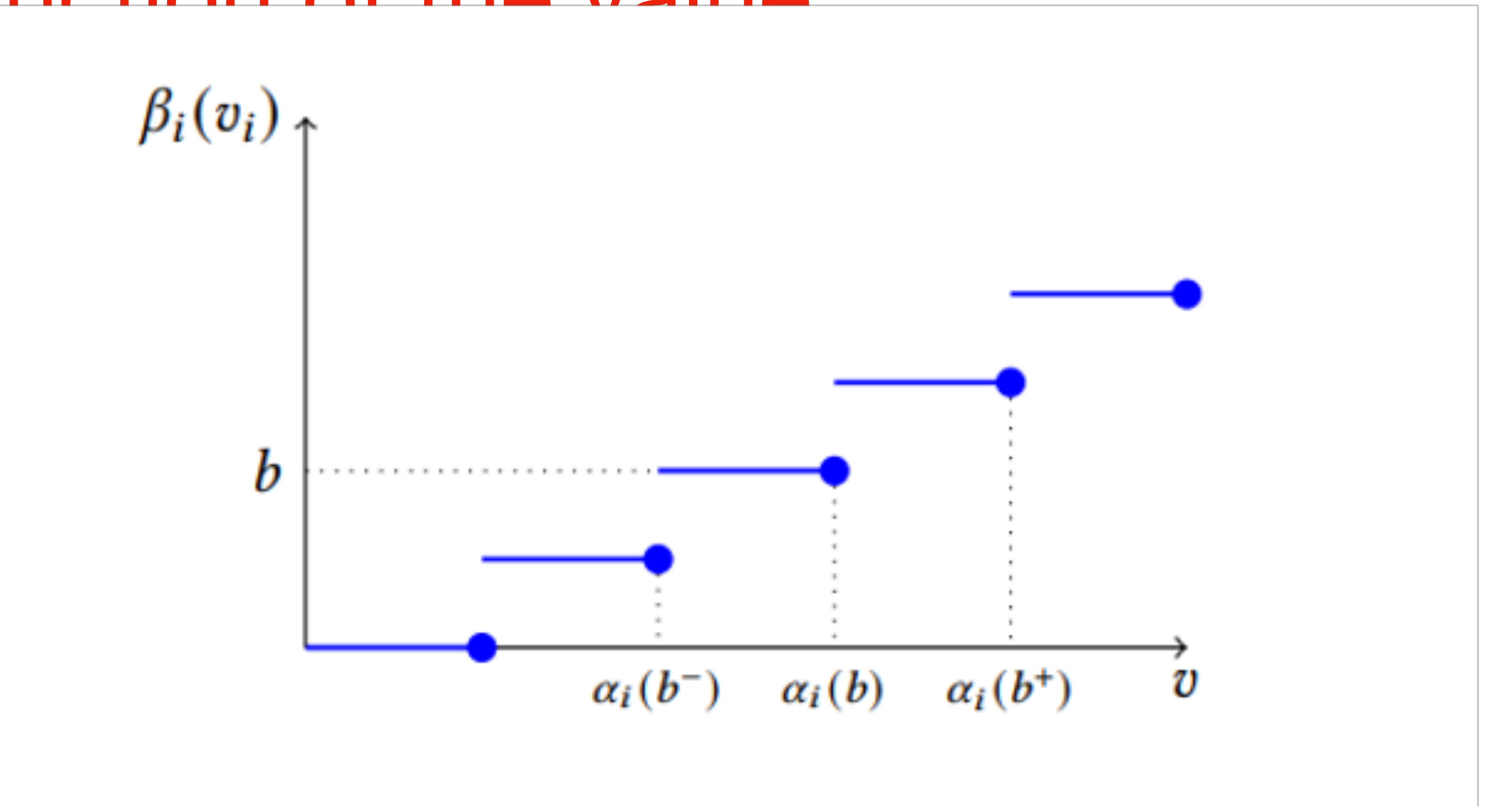

PPAD-completeness

- PPAD-membership: New proof of existence via Brouwer's fixed point theorem. Brouwer function is polynomially continuous.

- PPAD-hardness: Reduction from ε-Generalized Circuit, a known PPAD-complete problem [Chen, Deng, and Teng 2009, Rubinstein 2018].

  - In fact we first show that ε-Generalized Circuit is still PPAD-complete, even when restricted to a very small set of gates.

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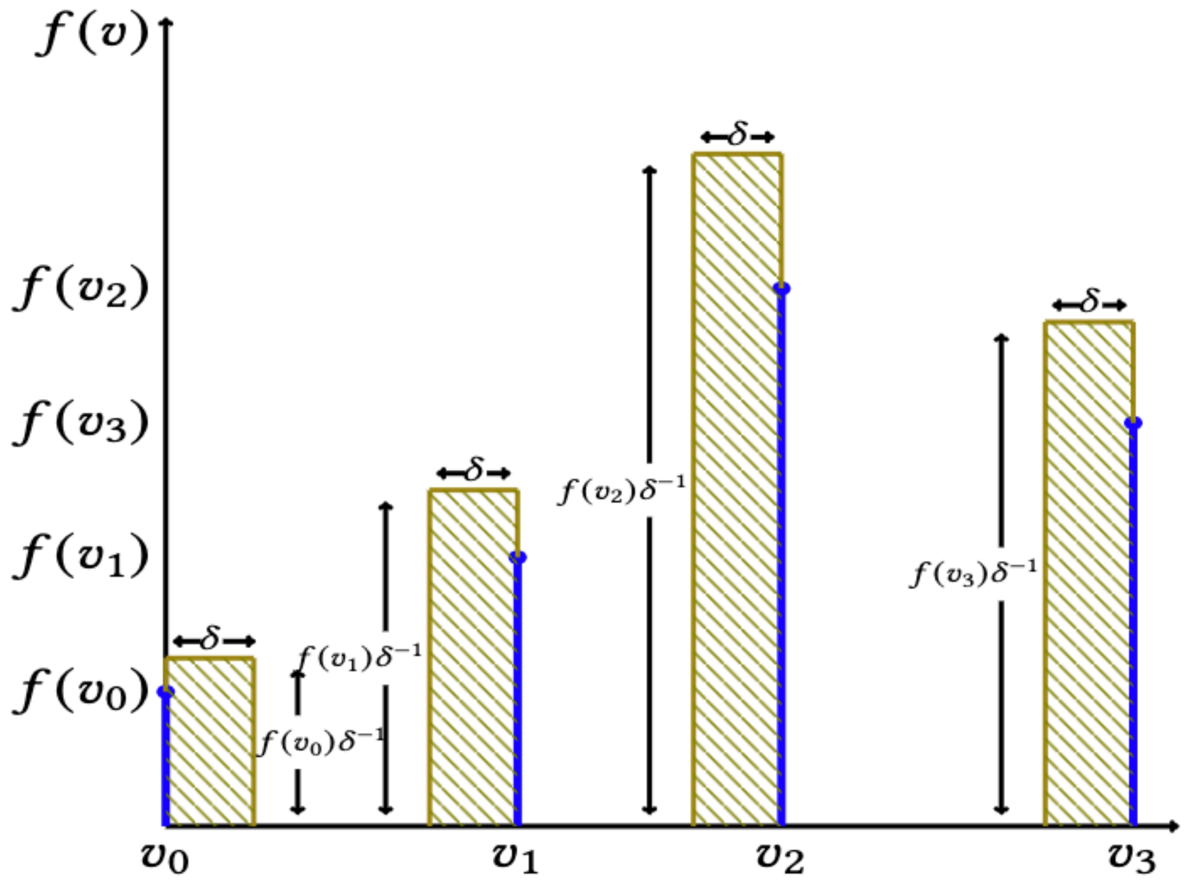

Figure 1: Discrete $\longrightarrow$ Continuous

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DFPA,
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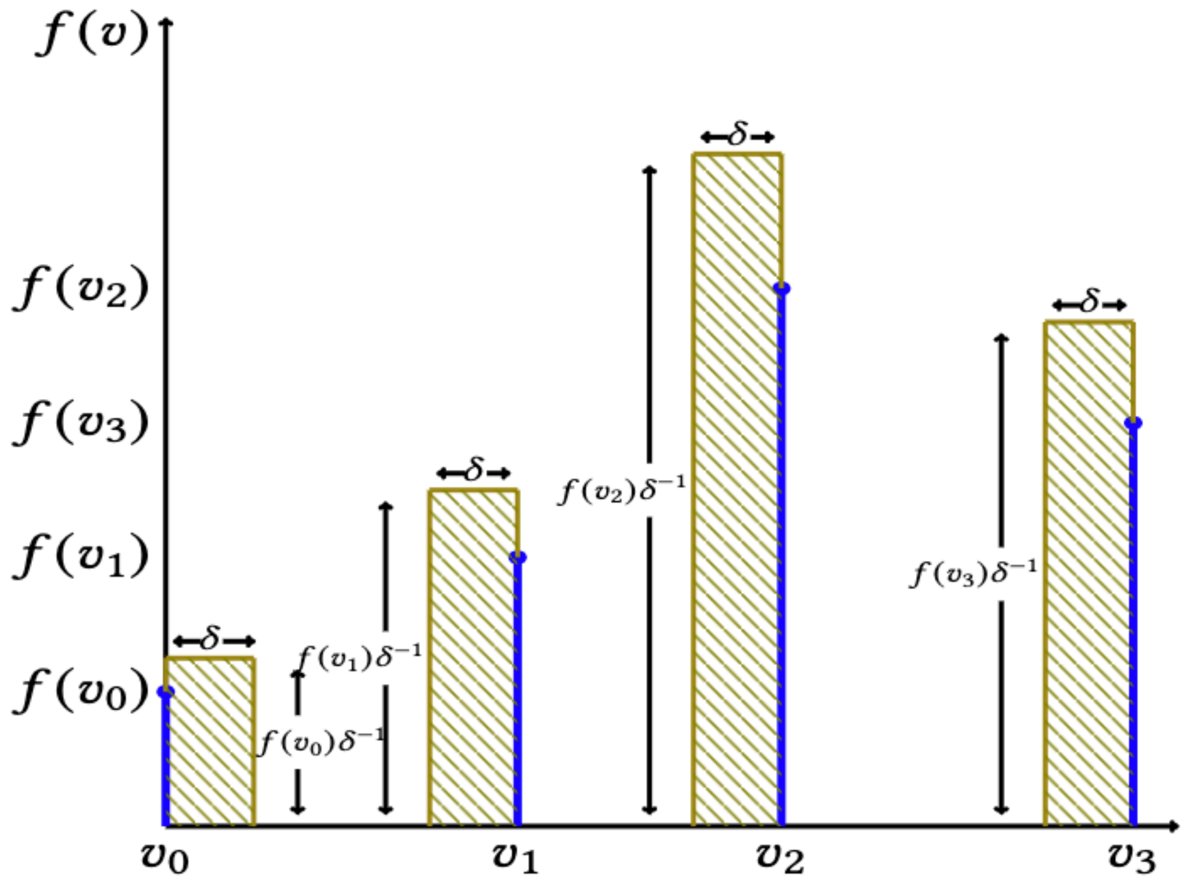

Figure 1: Discrete $\longrightarrow$ Continuous

CFPA

Equivalence Result

Discrete

Continuous

DFPA,
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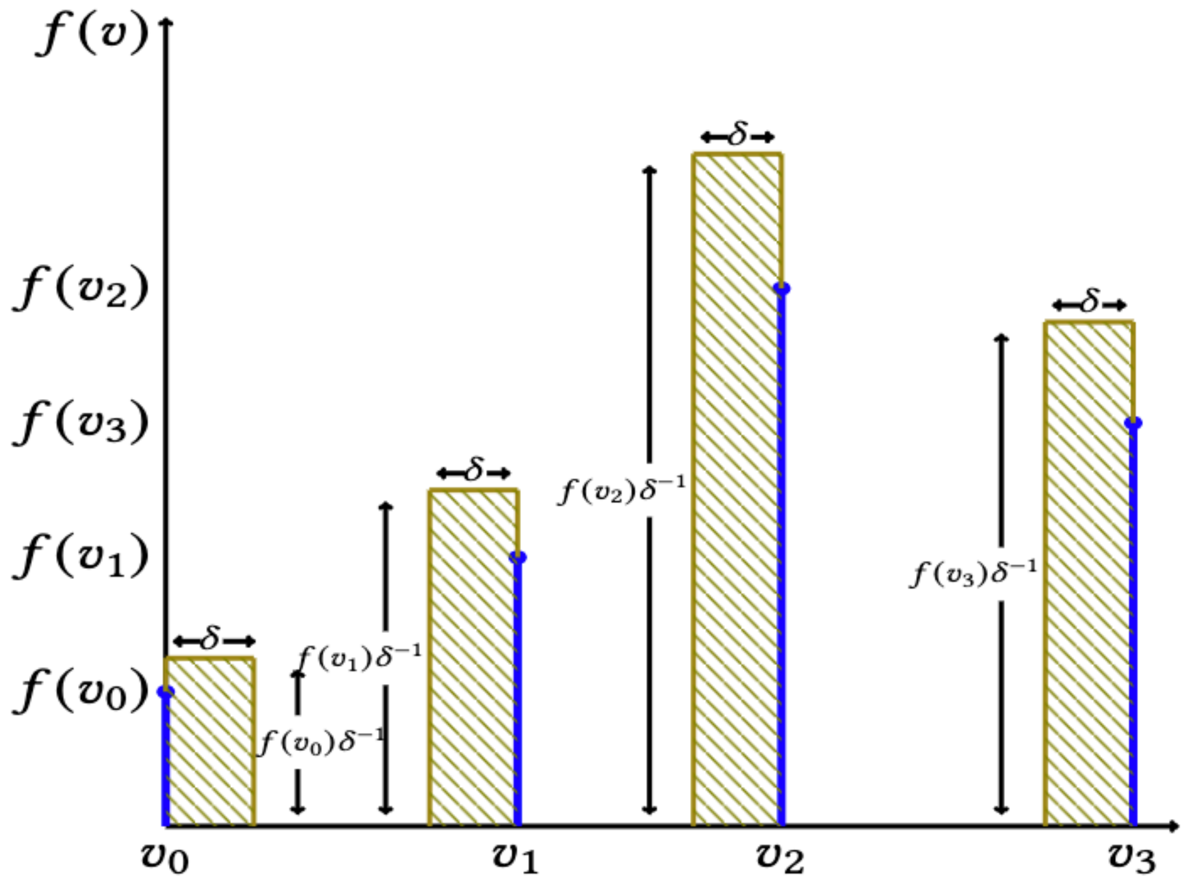

Figure 1: Discrete ⟶ Continuous

CFPA

ε-PBNE,
$\forall \varepsilon \geq 0$

Equivalence Result

Discrete

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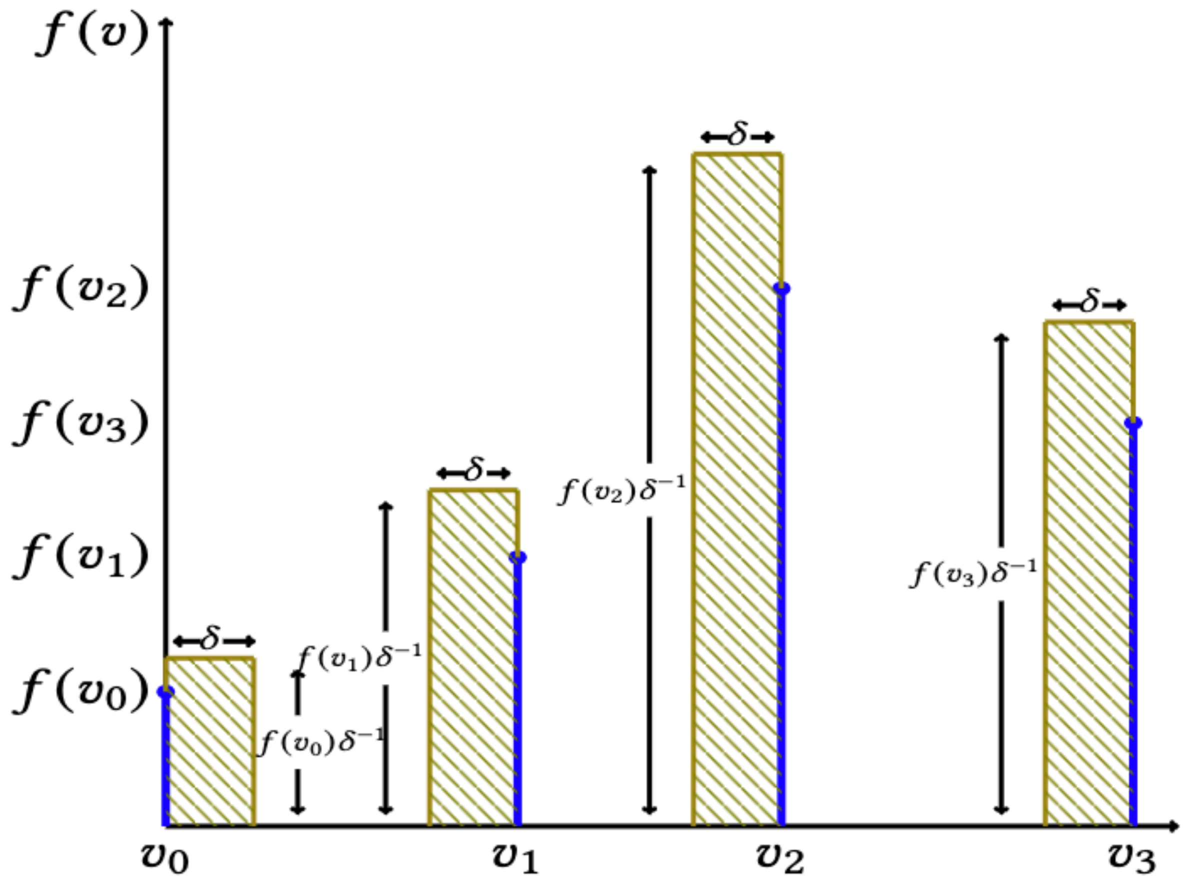

Figure 1: Discrete $\longrightarrow$ Continuous

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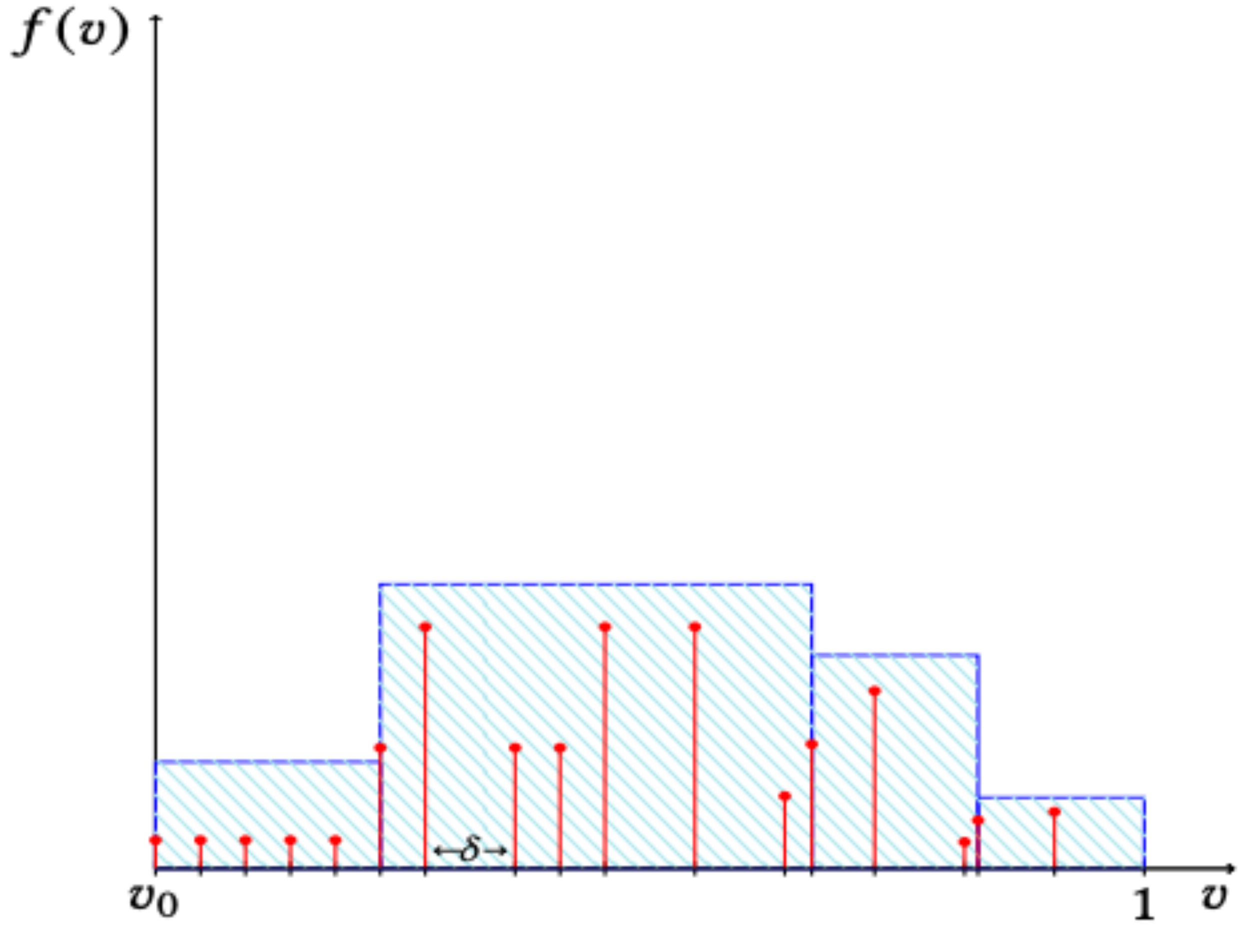

Figure 2: Continuous ⟶ Discrete

Equivalence Result

Continuous

Discrete

CFPA,
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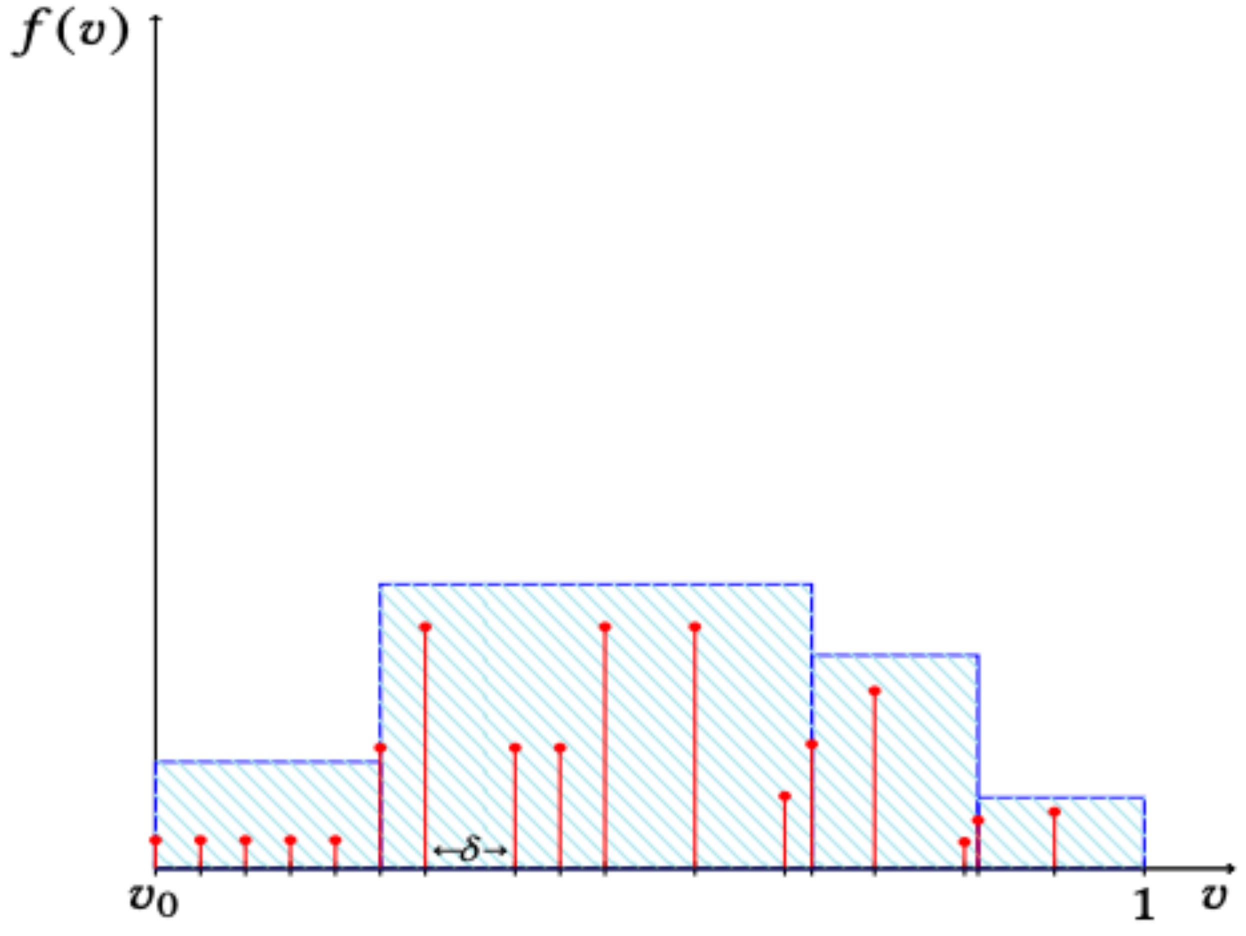

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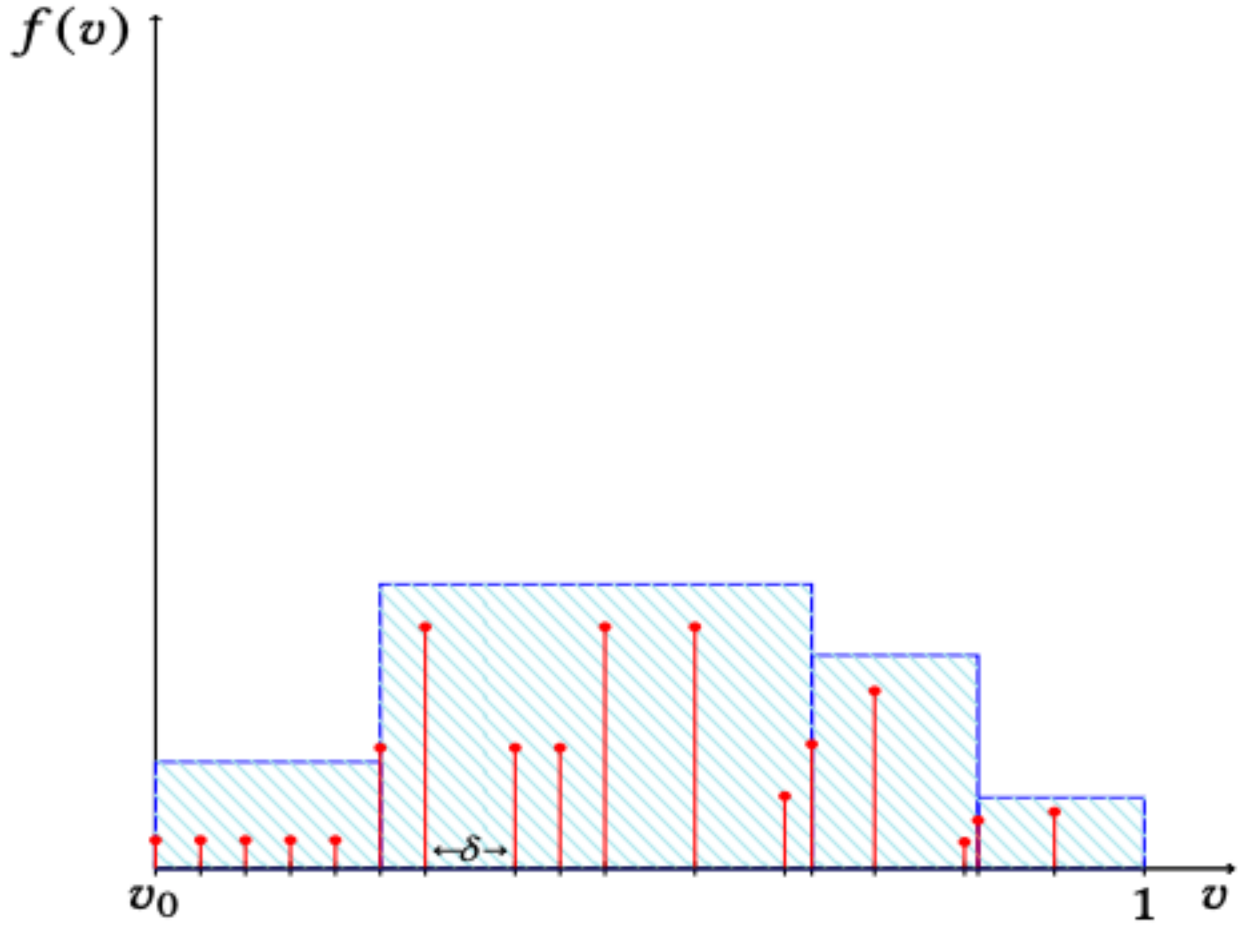

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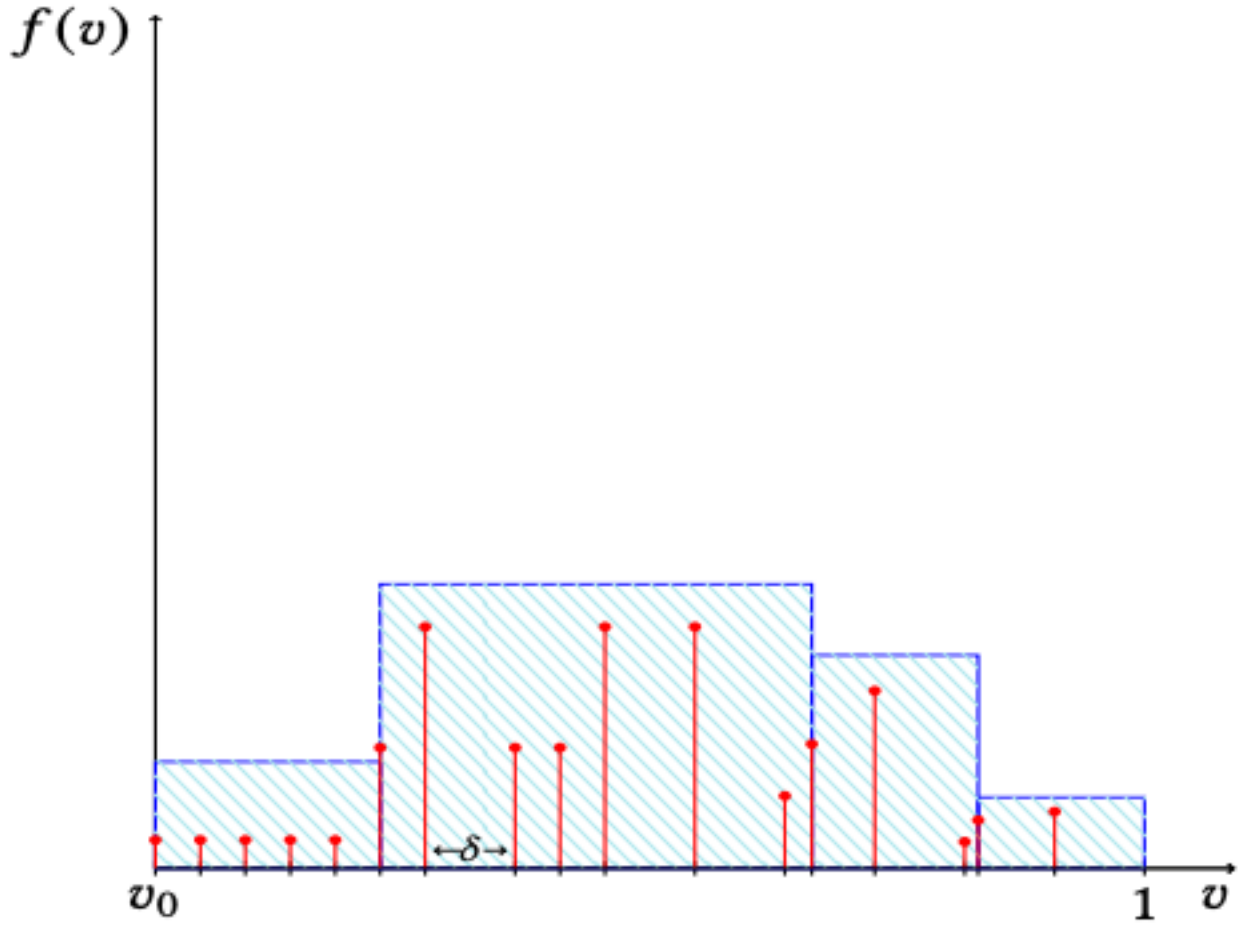

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PPAD-completeness

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- PPAD-hardness: Reduction from the PPAD-complete problem PURE-CIRCUIT [DFHM22].

The iid Setting

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- Polynomial Time Approximation Scheme (PTAS): An algorithm that computes an ε-approximate solution to a problem in time polynomial to the inputs, but possibly exponential in 1/ε.

Continuous Priors

Theorem [FGHLP23]: Computing an ε-PBNE with subjective priors is PPAD-complete.

Discrete Priors

Theorem [FGHK24]: Deciding the existence of an ε-PBNE with subjective priors is NP-complete.

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Theorem [FGHK24]: The problem of computing an ε-MBNE with iid priors admits a PTAS.

The iid Setting

Proof Sketch

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3. Round the solution achieved in Step 2 so that it corresponds to a feasible set of strategies, provide a bound on the approximation factor of the MBNE.

Complexity Landscape

MBNE: PPAD-complete [FGHK24]

PBNE: NP-complete [FGHK24]

discrete priors

iid

IPV

subjective priors

Complexity Landscape

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Complexity Landscape

continuous priors

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iid priors | IPV | subjective priors

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polytime	complete for other class	PPAD-complete