Alternation makes the adversary weaker

Stratis Skoulakis

joint work with

V. Cevher, **EPFL**

A. Cutkosky, **BU**

G. Piliouras,
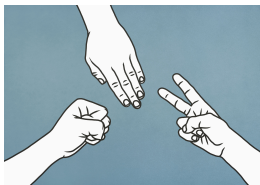**DeepMind**

A. Kavis, **UT Austin**

L. Viano, **EPFL**

Normal Form Games
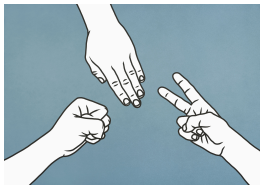
Rock-Paper-Scissors

Payoff matrix of Alice

Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Payoff matrix of Bob

Alice/Bob	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Normal Form Games

Rock-Paper-Scissors

Alice plays Rock

Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Alice gets -1

Bob plays Scissors

Alice/Bob	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Bob gets 1

Normal Form Games

Normal-Form Games

Matrices A and B

Payoff matrix A of Alice

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{12}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Payoff matrix B of Bob

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

Normal Form Games

Normal-Form Games

Matrices A and B

Alice plays action i

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{1j}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{nj}	$\cdots$	A_{nm}

Alice gets A_{ij}

Bob plays action j

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{1j}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{nj}	$\cdots$	B_{nm}

Bob gets B_{ij}

Normal Form Games

Normal-Form Games - Mixed Strategies

Matrices A and B

Alice plays a prob. distr.

$x = (x_1, \ldots, x_n) \in \Delta_n$

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{12}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Bob plays a prob. distr.

$y = (y_1, \ldots, y_m) \in \Delta_m$

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

Normal Form Games

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$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Alice's expected cost
$$x^\top A y$$

Bob plays a prob. distr.
$y = (y_1, \ldots, y_m) \in \Delta_m$

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

Bob's expected cost
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Normal Form Games

Normal-Form Games - Mixed Strategies

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$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
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Bob plays a prob. distr.
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$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

$$x^\top B y$$

Normal-Form Games over Time

- Alice and Bob play the normal-form game (A, B) over T rounds.

  ▶ *Simultaneous play*: Agents <u>simultaneously</u> select their strategies at each round (very well-studied).

  ▶ *Alternating play*: Agents <u>alternatingly</u> update their strategies (this work).

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects $y_1 \longrightarrow$ Alice gets $x_1^\top A y_1$

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1 $\longrightarrow$ Alice gets $x_1^\top A y_1$
- Alice selects x_2 / Bob selects y_2

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1 $\longrightarrow$ Alice gets $x_1^\top A y_1$
- Alice selects x_2 / Bob selects y_2 $\longrightarrow$ Alice gets $x_2^\top A y_2$

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1 $\longrightarrow$ Alice gets $x_1^\top A y_1$
- Alice selects x_2 / Bob selects y_2 $\longrightarrow$ Alice gets $x_2^\top A y_2$
- Alice selects x_3 / Bob selects y_3

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects $y_1 \longrightarrow$ Alice gets $x_1^\top A y_1$
- Alice selects x_2 / Bob selects $y_2 \longrightarrow$ Alice gets $x_2^\top A y_2$
- Alice selects x_3 / Bob selects $y_3 \longrightarrow$ Alice gets $x_3^\top A y_3$
- $\ldots$

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1 $\longrightarrow$ Alice gets $x_1^\top A y_1$
- Alice selects x_2 / Bob selects y_2 $\longrightarrow$ Alice gets $x_2^\top A y_2$
- Alice selects x_3 / Bob selects y_3 $\longrightarrow$ Alice gets $x_3^\top A y_3$
- . . .

How shoud Alice select her actions over time?

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

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- Alice selects x_2 / Bob selects y_2 $\longrightarrow$ Alice gets $x_2^\top A y_2$
- Alice selects x_3 / Bob selects y_3 $\longrightarrow$ Alice gets $x_3^\top A y_3$
- . . .

How shoud Alice select her actions over time? *No Regret algorithms*

- Regret Matching [Blackwell '65]
- Hedge [Freund et al. '97]
- Online Gradient Descent [Zinkevic '03]
- Follow the Regularized Leader [Abernethy et al. '10]
- ....

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- Alice selects x_2 / Bob selects $y_2 \longrightarrow$ Alice gets $x_2^\top A y_2$
- . . .

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- Alice selects x_2 / Bob selects $y_2 \longrightarrow$ Alice gets $x_2^\top A y_2$
- $\ldots$

Hedge Algorithm [Freund and Schapire '97] Godel Prize '03

$$x_{t+1}(i) = \frac{x_t(i) \cdot e^{-\gamma [Ay_t]_i}}{\sum_{j=1}^{n} x_t(j) \cdot e^{-\gamma [Ay_t]_j}}$$

Simultaneous play

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- ...

Hedge Algorithm [Freund and Schapire '97] Godel Prize '03

$$x_{t+1}(i) = \frac{x_t(i) \cdot e^{-\gamma [Ay_t]_i}}{\sum_{j=1}^{n} x_t(j) \cdot e^{-\gamma [Ay_t]_j}}$$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies $y_1, \ldots, y_T$, *the regret of Alice* $\mathcal{R}(T)$

$$\mathcal{R}(T) := \underbrace{\sum_{t=1}^{T} x_t^\top A y_t}_{\text{cost of Alice}} - \underbrace{\min_{i \in [n]} \sum_{t=1}^{T} [Ay_t]_i}_{\text{cost of best action}} \leq \tilde{\mathcal{O}}\left(\sqrt{T}\right)$$

Simultaneous play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects y_1 $\longrightarrow$ Alice gets $x_1^\top A y_1$
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- Alice selects x_3 / Bob selects y_3 $\longrightarrow$ Alice gets $x_3^\top A y_3$
- $\ldots$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies,

$$\underbrace{\frac{1}{T} \sum_{t=1}^{T} x_t^\top A y_t}_{\text{time-average cost}} \quad \leq \quad \underbrace{\frac{1}{T} \min_{x \in \Delta_n} \sum_{t=1}^{T} x^\top A y_t}_{\text{best fixed action}} \quad + \quad \tilde{\mathcal{O}}\left(\frac{\sqrt{T}}{T} \right) \to 0$$

Simultaneous play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects $y_1 \longrightarrow$ Alice gets $x_1^\top A y_1$
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- $\ldots$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies $y_1, \ldots, y_T$, the regret of Alice $\mathcal{R}(T)$

$$\mathcal{R}(T) \quad := \quad \underbrace{\sum_{t=1}^{T} x_t^\top A y_t}_{\textit{cost of Alice}} \quad - \quad \underbrace{\min_{i \in [n]} \sum_{t=1}^{T} [A y_t]_i}_{\textit{cost of best action}} \quad \leq \quad \tilde{\mathcal{O}}\left(\sqrt{T}\right)$$

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Can Alice do better?

Simultaneous play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

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- Alice selects x_2 / Bob selects $y_2 \longrightarrow$ Alice gets $x_2^\top A y_2$
- $\dots$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies $y_1, \dots, y_T$, the regret of Alice $\mathcal{R}(T)$

$$\mathcal{R}(T) \;\; \coloneqq \;\; \underbrace{\sum_{t=1}^{T} x_t^\top A y_t}_{\text{cost of Alice}} \;\; - \;\; \underbrace{\min_{i \in [n]} \sum_{t=1}^{T} [A y_t]_i}_{\text{cost of best action}} \;\; \leq \;\; \tilde{\mathcal{O}}\left(\sqrt{T}\right)$$

Can Alice do better? No!

Simultaneous play

Simultaneous Play

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- ...

Theorem (folkore)

*Bob **can always select** $y_1, \ldots, y_T$, the regret of Alice*

$$\mathcal{R}(T) \; := \; \underbrace{\sum_{t=1}^{T} x_t^\top A y_t}_{\text{cost of Alice}} \; - \; \underbrace{\min_{i \in [n]} \sum_{t=1}^{T} [A y_t]_i}_{\text{cost of best action}} \; \geq \; \Omega(\sqrt{T})$$

$\mathcal{O}(\sqrt{T})$ regret is the best Alice can get in **simultaneous play**!

Simultaneous play

Simultaneous Play

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What if Alice and Bob play in alternating turns?

Solving Heads'up Poker

Online Learning + Simultaneous Play

Polaris [Bowling et al. AAMAS '09] → Decent Performance

Solving Heads'up Poker

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Online Learning + Alternating Play

Cepheus [Oskari et al. IJCAI '15], Libratus [Brown et al. IJCAI '17] → **Beat Human Experts!**

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Cepheus [Oskari et al. IJCAI '15], Libratus [Brown et al. IJCAI '17] → **Beat Human Experts!**

Regret Guarantees → Faster Training!

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

Alternating Turns

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Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0
- Alice selects x_1

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0
- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0
- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$
- Bob selects y_2

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Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

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- Alice selects x_3

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- ...

Can Alice have regret $\mathcal{R}(T)$ better than $\mathcal{O}(\sqrt{T})$?

$$\mathcal{R}(T) := \underbrace{\sum_{k=0}^{T/2} x_{2k+1}^\top A(y_{2k} + y_{2k+2})}_{\text{Alice's cost}} - \underbrace{\min_{x \in \Delta_n} \sum_{k=0}^{T/2} x^\top A(y_{2k} + y_{2k+2})}_{\text{best fixed action}}$$

Alternating Turns

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- ...

Theorem ([Cevher, Cutksovsky*, Piliouras, Skoulakis*, Viano NeurIPS '23 spotlight])

In alternating play, Alice can always guarantee

- $\tilde{\mathcal{O}}(T^{1/3})$ regret for general games.
- $\mathcal{O}(\log T)$ for $n = 2$ actions (different algorithm).

Alternating Turns

Previous Results

- If **both** Alice and Bob use *Gradient Descent* in **unconstrained zero-sum games** $\longrightarrow \mathcal{O}(1)$ regret [Bailey et al. COLT 2020]

- If **both** Alice and Bob use *Hedge* in **zero-sum games** $\longrightarrow \mathcal{O}(T^{1/3})$ regret [Wibisono et al. NeurIPS 2022]

Alternating Turns

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Previous results make assumptions on Bob's behavior.

Alternating Turns

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Theorem ([Cevher, Cutksovsky*, Piliouras, Skoulakis*, Viano NeurIPS '23 spotlight])

In alternating play, *Alice can always guarantee*

- $\tilde{\mathcal{O}}(T^{1/3})$ *regret for general games.*
- $\mathcal{O}(\log T)$ *for $n = 2$ actions (different algorithm).*

no matter Bob's behavior!

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0
- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$
- Bob selects $y_2 \longrightarrow$ Alice gets $x_1^\top A y_2$
- Alice selects $x_3 \longrightarrow$ Alice gets $x_3^\top A y_2$
- ...

	Simultaneous Play	Alternating Play
General Games	$\tilde{\mathcal{O}}(\sqrt{T})$ tight	$\tilde{\mathcal{O}}(T^{1/3})$
2 actions for Alice	$\tilde{\mathcal{O}}(\sqrt{T})$ tight	$\tilde{\mathcal{O}}(\log T)$

Alternating Turns

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- Bob selects $y_2 \longrightarrow$ Alice gets $x_1^\top A y_2$
- Alice selects $x_3 \longrightarrow$ Alice gets $x_3^\top A y_2$
- . . .

Our algorithm (general simplex)

$$x_{2k+1} \leftarrow \operatorname{argmin}_{x \in \Delta_n} \left[3 \underbrace{x^\top A y_{2k}}_{\text{exploits } y_{2k}} + 2 \underbrace{\sum_{k'=0}^{k-1} x^\top A y_{2k'}}_{\text{reinforces good past actions}} - \underbrace{\gamma \cdot \sum_{i=1}^{n} \log x_i}_{\text{prevents overfitting}} \right]$$

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Setting $\gamma = \mathcal{O}(T^{1/3}) \to \mathcal{O}(T^{1/3})$ alternating regret

2 **action simplex**

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- Alice plays x_{2k+1} in the convex hull of z_k and w_k.

Take-Away Message

Algorithms beyond $\mathcal{O}(\sqrt{T})$ regret lower bounds of simultaneous play.

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Thank you!!

References