On the Complexity of Higher Order Abstract Voronoi Diagrams⋆

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Abstract. Abstract Voronoi diagrams [17, 18] are based on bisecting curves enjoying simple combinatorial properties, rather than on the geometric notions of sites and circles. They serve as a unifying concept. Once the bisector system of any concrete type of Voronoi diagram is shown to fulfill the AVD properties, structural results and efficient algorithms become available without further effort. For example, the first optimal algorithms for constructing nearest Voronoi diagrams of disjoint convex objects, or of line segments under the Hausdorff metric, have been obtained this way [22].
In a concrete order-k Voronoi diagram, all points are placed into the same region that have the same k nearest neighbors among the given sites. This paper is the first to study *abstract Voronoi diagrams of arbitrary order k*. We prove that their complexity is upper bounded by $2k(n-k)$. So far, an $O(k(n-k))$ bound has been shown only for point sites in the Euclidean and L_p plane [20, 21], and, very recently, for line segments [25]. These proofs made extensive use of the geometry of the sites.
Our result on AVDs implies a $2k(n-k)$ upper bound for a wide range of cases for which only trivial upper complexity bounds were previously known, and a slightly sharper bound for the known cases.
Also, our proof shows that the reasons for this bound are combinatorial properties of certain permutation sequences.
Keywords: Abstract Voronoi diagrams, computational geometry, distance problems, higher order Voronoi diagrams, Voronoi diagrams.

1 Introduction

Voronoi diagrams are useful structures, known in many areas of science. The underlying idea goes back to Descartes [13]. There are sites p, q that exert influence on their surrounding space, M. Each point of M is assigned to that site p (resp. to those sites $p_1, \ldots, p_k$) for which the influence is strongest. Points assigned to the same site(s) form *Voronoi regions.*

The nature of the sites, the measure of influence, and space M can vary. The *order*, k, can range from 1 to $n-1$ if n sites are given. For $k = 1$ the standard

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nearest Voronoi diagram results, while for $k = n-1$ the *farthest* Voronoi diagram is obtained, where all points of M having the same farthest site are placed in the same Voronoi region. In this paper we are interested in values of k between 1 and $n-1$; here an *order-k* Voronoi region contains all points that have the same k nearest sites. See the surveys and monographs [6, 8, 9, 11, 14, 24].

A lot of attention has been given to nearest Voronoi diagrams in the plane. Many concrete cases have the following features in common. The locus of all points at identical distance to two sites p, q is an unbounded curve $J(p, q)$. It bisects the plane into two domains, $D(p, q)$ and $D(q, p)$; domain $D(p, q)$ consists of all points closer to p than to q. Intersecting all $D(p, q)$, where $q \neq p$ for a fixed p, results in the Voronoi region $\mathrm{VR}(p, S)$ of p with respect to site set S. It equals the set of all points with unique nearest neighbor p in S. If geodesics exist, Voronoi regions are pathwise connected, and the union of their closures covers the plane, since each point has at least one nearest neighbor in S.

In *abstract Voronoi diagrams* (AVDs, for short) no sites or distance measures are given. Instead, one takes unbounded curves $J(p, q) = J(q, p)$ as primary objects, together with the domains $D(p, q)$ and $D(q, p)$ they separate. *Nearest abstract Voronoi regions* are defined by

$$\mathrm{VR}(p, S) := \bigcap_{q \in S \setminus \{p\}} D(p, q),$$

and now one *requires* that the following properties hold true for each subset S' of S.

(A1) Each nearest Voronoi region $VR(p, S')$ is pathwise connected.
(A2) Each point of the plane belongs to the closure of a nearest Voronoi region $VR(p, S')$.

Two more, rather technical, assumptions on the curves $J(p, q)$ are stated in Definition 1 below. It has been shown that the resulting nearest AVDs—the plane minus all Voronoi regions— are planar graphs of complexity $O(n)$. They can be constructed, by randomized incremental construction, in $O(n \log n)$ many steps [18, 19, 22]. Moreover, properties (A1) and (A2) need only be checked for all subsets S' of size three [18]. This makes it easier to verify that a concrete Voronoi diagram is under the roof of the AVD concept. Examples of such applications can be found in [1–3, 10, 16, 22]. *Farthest abstract Voronoi diagrams* consist of regions $\mathrm{VR}^*(p, S) := \bigcap_{q \in S \setminus \{p\}} D(q, p)$. They have been shown to be trees of complexity $O(n)$, computable in expected $O(n \log n)$ many steps [23].

In this paper we consider, for the first time, general *order-k abstract Voronoi regions*, defined by

$$\mathrm{VR}^k(P, S) := \bigcap_{p \in P,\ q \in S \setminus P} D(p, q),$$

for each subset P of S of size k. The order-k abstract Voronoi diagram $V^k(S)$ is defined to be the complement of all order-k Voronoi regions in the plane; it equals the collection of all edges that separate order-k Voronoi regions (Lemma 4). In addition to properties (A1) and (A2) we shall assume the following.

(A3) No nearest Voronoi region $VR(p, S')$ is empty.

In Lemma 1 we prove that property (A3) needs only be tested for all subsets S' of size four. Clearly, (A3) holds in all concrete cases where each nearest region contains its site.

Figure 1 shows two concrete order-2 diagrams of points and line segments under the Euclidean metric. We observe that the order-2 Voronoi region of line segments s_1, s_2 is disconnected. In general, a Voronoi region in $V^2(S)$ can have $n-1$ connected components. Figure 2 depicts a curve system fulfilling all properties required, and the resulting abstract Voronoi diagrams of order 1 to 4. An index p placed next to a curve indicates on which side the domain $D(p,q)$ lies, if q denotes the opposite index. The order-2 region of p_1 and p_2 consists of four connected components.

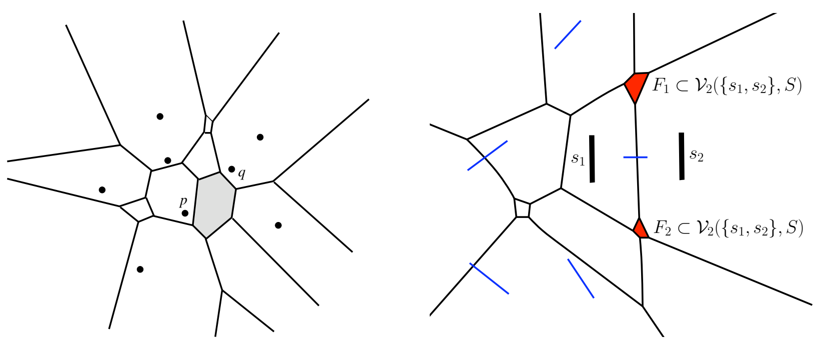


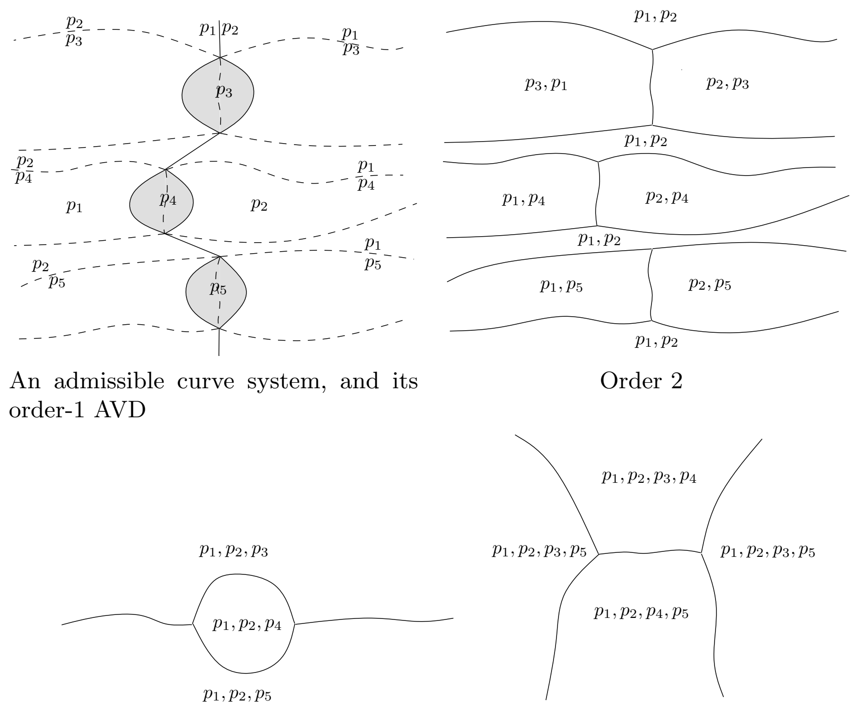
Fig. 1: Order-2 diagrams of points and line segments

In this paper we are proving the following result.

Theorem 1. *The abstract order-k Voronoi diagram $V^k(S)$ has at most $2k(n-k)$ many faces.*

So far, an $O(k(n-k))$ bound was known only for points in L_2 and in the L_p-plane [20,21]. Quite recently, it has been shown for line segments in the Euclidean plane [25], too. The proofs of these results depend on geometric arguments using k-sets[3], k-nearest neighbor Delaunay triangulations, and point-line duality, respectively. None of these arguments applies to abstract Voronoi diagrams.

[3] We call a subset of size k of n points a *k-set* if it can be separated by a line passing through two other points. Such k-sets correspond to unbounded order-$(k+1)$ Voronoi edges.



An admissible curve system, and its order-1 AVD

Order 2

Order 3

Order 4

Fig. 2: AVD of 5 sites in all orders.

However, the upper bound on k-sets established in [5] had a combinatorial proof; it was obtained by analyzing the cyclic permutation sequences that result when projecting n point sites onto a rotating line. In such a sequence, consecutive permutations differ by a switch of adjacent elements, and permutations at distance $\binom{n}{2}$ are inverse to each other.

In this paper we traverse the unbounded edges of higher order AVDs, and obtain a strictly larger class of cyclic permutation sequences, where consecutive permutations differ by switches and any two elements switch exactly twice. Our proof is based on a tight upper bound to the number of switches that can occur among the first $k+1$ elements; see Lemma 9. It is interesting to observe that in our class, each permutation sequence can be realised by an AVD (Lemma 10), while a similar statement does not hold for the sequences obtained by point projection [15].

To avoid technical complications we are assuming, in this paper, that any two input curves $J(p, q)$ intersect in a finite number of points, and that Voronoi vertices are of degree 3. How to get rid of the first assumption has been shown for the case of nearest AVDs in [18]. The perturbation technique of [17] can be used to obtain degree 3 vertices.

Theorem 1 implies a $2k(n-k)$ upper complexity bound on a wide range order-k Voronoi diagrams for which no good bounds were previously known. For example, sites may be disjoint convex objects of constant complexity in L_2 or under the Hausdorff metric. For point sites, distance can be measured by any metric d satisfying the following conditions: points in general position have unbounded bisector curves; d-circles are of constant algebraic complexity; each d-circle contains an L_2-circle and vice versa; for any two points $a \neq c$ there is a third point $b \neq a, c$ such that $d(a,c) = d(a,b) + d(b,c)$ holds. This includes all convex distance functions, but also the Karlsruhe metric where motions are constrained to radial or circular segments with respect to a fixed center point. A third example are point sites with additive weights a_p, a_q that satisfy $|a_p - a_q| < |p - q|$, for any two sites $p \neq q$; see [8] for a discussion of these examples.

The rest of this paper is organized as follows. In Section 2 we present some basic facts about AVDs. Then, in Section 3, permutation sequences will be studied, in order to establish an upper bound to the number of unbounded Voronoi edges of order at most k. This will lead, in Section 4, to a tight upper bound for the number of faces of order k.

2 Preliminaries

In this section we present some basic facts on abstract Voronoi diagrams of various orders.

Definition 1. *A curve system $J := \{J(p,q) : p \neq q \in S\}$ is called* admissible *if it fulfills, besides axioms (A1), (A2), (A3) stated in the introduction, the following axioms.*

(A4) Each curve $J(p,q)$, where $p \neq q$, is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole.

(A5) Any two curves $J(p,q)$ and $J(r,t)$ have only finitely many intersection points, and these intersections are transversal.

Fortunately, verification of these axioms can be based on constant size examples.

Lemma 1. *To verify axioms (A1) and (A2) it is sufficient to check all subsets S' of size 3, and for (A3), of size 4.*

Proofs for (A1) and (A2) can be found in [18], Section 4.3. For (A3), the proof is given in the Appendix.

The following fact will be very useful in the sequel. Its proof can be found in [18], Lemma 5.

Lemma 2. *For all p, q, r in S, $D(p,q) \cap D(q,r) \subseteq D(p,r)$ holds.*

Consequently, for each $x \notin \bigcup_{p,q \in S} J(p,q)$ a global ordering of the site set S is given by

$$p <_x q :\Longleftrightarrow x \in D(p,q).$$

Informally, one can interpret $p <_x q$ as "x is closer to p than to q". We will write $p < q$ if it is clear which $x \in \mathbb{R}^2$ we are referring to.

As a direct consequence we show that property (A2) also holds for abstract order-k Voronoi regions.

Lemma 3. *Let $J = \{J(p,q) : p \neq q \in S\}$ be an admissible curve system. Then for all $k \in \{1, \ldots n-1\}$*

$$\mathbb{R}^2 = \bigcup_{P \subseteq S, |P|=k} \overline{VR^k(P,S)}.$$

Proof. Let $x \in \mathbb{R}^2$. If x is not contained in any bisecting curve $J(p,q)$ then it belongs to the order-k region $\mathrm{VR}^k(P,S)$, where $P = \{p_1, \ldots, p_k\}$ are the k smallest elements of S with respect to the ordering $<_x$. Otherwise, x lies on the boundary of a domain $D \subset \mathbb{R}^2 \setminus \bigcup_{p \neq q \in S} J(p,q)$, and D fully belongs to an order-k region.

The proofs of the following Lemmata 4 and 5 are similar to the proof of Lemma 3.

Lemma 4.

$$V^k(S) = \bigcup_{\substack{P \neq P' \subset S \\ |P|=|P'|=k}} \overline{VR^k(P,S)} \cap \overline{VR^k(P',S)}$$

Lemma 5. *If the intersection $E := \overline{VR^k(P,S)} \cap \overline{VR^k(P',S)}$ is not empty, there are sites $p \in P$ and $p' \in P'$ such that $P \setminus \{p\} = P' \setminus \{p'\}$, and $E \subseteq J(p,p')$ holds. For each point $x \in VR^k(P,S)$ near E, index p is the k-th with respect to $<_x$, while for points x' in $VR^k(P',S)$ index p' appears at position k.*

In particular, $D(p,p')$ lies on the same side of $J(p,p')$ as $\mathrm{VR}^k(P,S)$ does.

If F, F' are connected components (*faces*) of $\mathrm{VR}^k(P,S)$ and $\mathrm{VR}^k(P',S)$, respectively, the intersection $\overline{F} \cap \overline{F'}$ can be empty, or otherwise be of dimension 0 (*Voronoi vertices*) or 1 (*Voronoi edges*).

For the next lemma we assume that all vertices are of degree 3. As in concrete order-k Voronoi diagrams [20] there are two types of vertices that can be distinguished by the nature of sets $P_1, P_2, P_3 \subset S$ which define the adjacent order-k Voronoi regions. In the first case there exist a set $H \subset S$ of size $k-1$ and three more indices $p, q, r \in S$ satisfying

$$P_1 = H \cup \{p\}, \quad P_2 = H \cup \{q\}, \quad P_3 = H \cup \{r\};$$

a vertex where such regions $\mathrm{VR}^k(P_i,S)$ meet is called *new* in $V^k(S)$, or *of nearest type*. In the second case, there are a subset $K \subset S$ of size $k-2$ and three more sites $p, q, r \in S$ such that

$$P_1 = K \cup \{p,q\}, \quad P_2 = K \cup \{p,r\}, \quad P_3 = K \cup \{q,r\}.$$

A vertex adjacent to such regions is called *old* in $V^k(S)$, or *of furthest type*. The proof of the following lemma follows quite directly from these definitions.

Lemma 6. *Let v be a new vertex in $V^k(S)$. Then v is an old vertex of $V^{k+1}(S)$, and v lies in the interior of a face of $V^{k+2}(S)$, i. e., v is no vertex of $V^{k+2}(S)$. Furthermore, every edge of $V^k(S)$ is included in a face of $V^{k+1}(S)$.*

Already in [23] it has been shown that farthest abstract Voronoi diagrams are trees, under a slightly different definition of admissible curves. In the Appendix we give a short alternative proof of this fact based on our axioms (A1)–(A5).

Lemma 7. *The farthest abstract Voronoi diagram $V^*(S)$ is a tree.*

3 Bounding the number of unbounded edges of $V^{\leq k}(S)$

Let Γ be a circle in $\mathbb{R}^2$ large enough such that no pair of bisectors cross on or outside of Γ (axiom (A5)).
If we walk around Γ the ordering $<_x$ on S changes whenever we cross a bisector $J(p, q)$. Here indices p and q switch their places in the ordering, and because of axiom (A5) there can be only one switch at a time. This means that each pair of sites switch exactly two times while walking one round around Γ, resulting in $n(n-1)$ switches altogether.

Lemma 8. *Suppose that two sites p and q switch in the ordering. Then they are adjacent to each other just before and after the switch.*

Proof. Let $p_1 < \ldots < p_n$, and assume that we cross $J(p_i, p_j)$, $i < j$, which means that p_i and p_j switch their places in the ordering. Suppose $j > i + 1$; then $x \in D(p_{i+1}, p_j)$ before the switch and $x \in D(p_j, p_{i+1})$ after the switch, but $J(p_{i+1}, p_j)$ has not been crossed—a contradiction.

Every time a switch among the first $k + 1$ elements of the ordering occurs, there is an unbounded edge of a Voronoi diagram of order $\leq k$. This means that *the maximum number of unbounded edges of all diagrams of order $\leq k$ is equal to the maximum number of switches among the first $k+1$ elements in the ordering.*

Permutation sequences and estimates for the maximum number of switches among the first k elements have been used in [5] to bound the number of k-sets of n points in the plane. These sequences resulted from projecting n points in general position onto a rotating line. Hence, they were of length $2N$, where $N = \binom{n}{2}$, and they had the following properties. Adjacent permutations differ by a transposition of adjacent elements, and any two permutations a distance N apart are inverse to each other. It has been shown in [15] that not every permutation of this type can be realized by a point set.

In the following lemma we introduce a larger class of permutation sequences that fits the AVD framework.

Lemma 9. *Let $P(S)$ be a cyclic sequence of permutations $P_0, \ldots, P_N = P_0$ such that*

(i) P_{i+1} *differs from* P_i *by an adjacent switch;*
(ii) *each pair of sites* $p, q \in S$ *switches exactly two times in* $P(S)$.

Then the number of switches occuring in $P(S)$ *among the first* $k+1$ *sites is upper bounded by* $k(2n-k-1)$. *Furthermore, this bound is tight.*

Proof. Call a switch *good* if it involves at least one of the k first sites of a permutation; otherwise call it *bad*. Consider the permutation $P_0 = (p_1, p_2, \ldots, p_n)$. For $i \in \{k+2, \ldots, n\}$, define B_i as the set of bad switches where p_i is switching with a site in $\{p_1, \ldots, p_{i-1}\}$. We remark that the sets B_i, for $i \in \{k+2, \ldots, n\}$, are pairwise disjoint. If p_i is not involved in a good switch, then all its $2i-2$ switches with sites in $\{p_1, \ldots, p_{i-1}\}$ are bad. Otherwise, for p_i to be involved in a good switch, it must first be involved in at least $i-k-1$ bad switches with sites in $\{p_1, \ldots, p_{i-1}\}$, in order to reach the first $k+1$ positions of a permutation, and since $P_0 = P_N$, it has to be involved in as many bad switches in order to return to its original place in P_N. In both cases, $|B_i| \geq 2(i-k-1)$. Because of (ii), the total number of switches is $N = 2\binom{n}{2}$. Therefore the number of good switches is at most

$$2\binom{n}{2} - \sum_{i=k+2}^{n} |B_i| \leq 2\binom{n}{2} - 2\sum_{j=1}^{n-k-1} j = k(2n-k-1),$$

where $j = i-k-1$.

To show that the bound is tight, let $P_0 = (p_1, \ldots, p_n)$. We will switch each p_i with all p_j having a place before p_i in P_0 in consecutive order until p_i is the first element and then in inverse order back. Start with $i=2$ and continue until $i=n$. Then the number of switches among the first $k+1$ sites is exactly $2\binom{n}{2} - 2\sum_{j=1}^{n-k-1} j$.

In contradistinction to the result in [15], each such permutation sequence can be realized by an AVD. The following Lemma 10 will be used for proving that the upper bound shown in Lemma 11 is tight. The proof of Lemma 10 is given in the Appendix.

Lemma 10. *Let* $P(S)$ *be a sequence of permutations as in Lemma 9. Then there exists an abstract Voronoi diagram where the ordering of the sites along* Γ *changes according to* $P(S)$.

Let S_i be the number of unbounded edges in $V^i(S)$. If an edge e has got two unbounded endpieces, i. e. the edge e bounding a p- and q-region is the whole bisector $J(p,q)$, then e is counted twice as an unbounded edge.

Lemma 11. *Let* $k \in \{1, \ldots, n-1\}$. *Then,*

$$k(k+1) \leq \sum_{i=1}^{k} S_i \leq k(2n-k-1).$$

Both bounds can be attained.

Proof. The second bound follows directly from Lemma 9. The first bound follows from the fact that the minimum number of switches among the first $(k+1)$ sites is greater or equal to the total number of switches, $n(n-1)$, minus the maximum number of switches among the last $(n-k)$ sites, which again is equal to the maximum number of switches among the first $(n-k)$ sites. Using Lemma 9 this implies

$$\sum_{i=1}^{k} S_i \geq n(n-1) - (n-k-1)(2n-(n-k-1)-1) = k(k+1).$$

The tightness of the bounds follows from Lemma 10.

4 Bounding the number of faces of $V^k(S)$

In the following we assume that each Voronoi vertex is of degree 3. The following two lemmata give combinatorial proofs for facts that were previously shown by geometric arguments [20, 25].

Lemma 12. *Let F be a face of $VR^{k+1}(H,S)$. The portion of $V^k(S)$ enclosed in F is exactly the farthest Voronoi diagram $V^*(H)$ intersected with F.*

Proof. "$\Rightarrow$": Let $x \in F$ and $x \in \mathrm{VR}^k(H',S)$. Since $F \subseteq \mathrm{VR}^{k+1}(H,S)$ it follows that $x \in D(p,q)$ for all $p \in H$ and $q \in S\backslash H$, implying $H' \subset H$. Let $H\backslash H' = \{r\}$, then $x \in D(p,r)$ for all $p \in H'$ and hence $x \in \mathrm{VR}^*(r,H)$.
"$\Leftarrow$": Let $x \in F$ and $x \in \mathrm{VR}^*(r,H)$. Then $x \in D(p,q)$ for all $p \in H$ and $q \in S\backslash H$ and $x \in D(p,r)$ for all $p \in H \setminus \{r\}$. This implies $x \in \mathrm{VR}^k(H \setminus \{r\}, S)$.

Lemma 13. *Let F be a face of $VR^k(H,S)$, $H \subseteq S$, $|H| = k \geq 2$. Then $V^*(H) \cap F$ is a nonempty tree.*

Proof. First we show that $\mathrm{V}^*(H) \cap F$ is not empty by assuming the opposite. Then there is a $p \in H$ such that $F \subseteq \mathrm{VR}^*(p,H)$. Let $F' \subseteq \mathrm{VR}^k(H',S)$ be a face of $V^k(S)$ adjacent to F along an edge e. By Lemma 5, we have $H = U \cup \{q\}$ and $H' = U \cup \{q'\}$, where q, q' are different and not contained in U. Also, $e \subseteq J(q,q')$ holds. If p were in U, we would obtain $F' \subseteq D(p,q)$ and $F \subseteq V^*(p,H) \subseteq D(q,p)$, hence $e \subseteq J(p,q)$—a contradiction to axiom (A5). Thus, $p \notin U$, which means $p = q$. Now Lemma 5 implies that each edge on the boundary of F has to be part of a curve $J(p,q_j)$ such that $D(p,q_j)$ lies on the F-side. Let $q_1,\ldots,q_i$ be the sites for which there is such an edge e on the boundary of F. Then $\mathrm{VR}^1(p,\{p,q_1,\ldots,q_i\}) = F$, because nearest Voronoi regions are connected thanks to axiom (A1). But from $F \subseteq V^*(p,H)$ it follows that $\mathrm{VR}^1(p,H) \subseteq \mathbb{R}^2 \setminus F$ and hence $\mathrm{VR}^1(p,S) \subseteq F \cap \mathbb{R}^2 \setminus F = \emptyset$, a contradiction to axiom (A3).

Next we show that $V^*(H) \cap F$ is a tree. Because of Lemma 7 it is clear that it is a forest. So it remains to prove that it is connected. Otherwise, there would be a domain $D \subset F$, bounded by two paths $P_1, P_2 \subset F$ of $V^*(H)$ and two disconnected parts e_1 and e_2 on the boundary of F. There is an index $p \in H$ such that $D \subseteq \mathrm{VR}^*(p, H)$. Since $V^*(H)$ is a tree, by Lemma 7, the upper (or: the lower) two endpoints of P_1 and P_2 must be connected by a path P in $V^*(H)$ that belongs to the boundary of $\mathrm{VR}^*(p, H)$; see Figure 3. Here path P connects the endpoints of e_1; both curves together encircle a domain D', which is part of $\mathrm{VR}^*(p, H)$. By definition of the farthest Voronoi diagram, there are $q_1, \ldots, q_i$, such that $e_1 \cup P$ is part of $J(p, q_1), \ldots, J(p, q_i)$, and all $D(p, q_j)$ are situated outside of D'; compare Lemma 5. But then $\mathrm{VR}^*(p, \{p, q_1, \ldots, q_i\})$ would be bounded, a contradiction to Lemma 7.

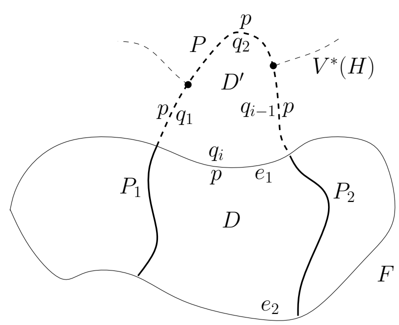


Fig. 3: The intersection of an order-k face F and the farthest Voronoi diagram of its defining sites must be a tree.

Lemma 14. *Let F be a face of $VR^{k+1}(H, S)$ and m the number of Voronoi vertices of $V^k(S)$ enclosed in its interior. Then F encloses $e = 2m + 1$ Voronoi edges of $V^k(S)$.*

Proof. Lemmata 12 and 13.

The next two lemmata are from [25]. The proofs are given in the Appendix, for completeness.

Lemma 15. *Let F_k, E_k, V_k and S_k denote, respectively, the number of faces, edges, vertices, and unbounded edges in $V^k(S)$. Then,*

$$E_k = 3(F_k - 1) - S_k \tag{1}$$

$$V_k = 2(F_k - 1) - S_k. \tag{2}$$

Lemma 16. *The number of faces in an AVD of order k is*

$$F_k = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i.$$

Theorem 2. *The number of faces F_k in an AVD of order k is bounded by bounds*

$$n - k + 1 \leq F_k \leq 2k(n - k) + k + 1 - n \in O(k(n - k)).$$

Both bounds can be attained.

Proof. Lemma 11 implies tight bounds $k(k-1) \leq \sum_{i=1}^{k-1} S_i \leq (k-1)(2n-k)$. Together with Lemma 16 this proves the theorem.

5 Concluding remarks

A natural question is if weaker axioms than (A1)–(A5) can still imply Theorem 2. In the case of nearest abstract Voronoi diagrams, it could be shown, with some technical effort, that (A5) is dispensable [18]. A big challenge will be to design an efficient algorithm for constructing abstract Voronoi diagrams of order k. Even for the special case of points in the Euclidean metric, no optimal algorithm is known for computing a single higher order Voronoi diagram.

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6 Appendix

Proof. (**of Lemma 1**) For (A1) and (A2) see [18], Section 4.3. If all bisecting curves are straight lines, (A3) needs only be verified for all subsets of size 3, by Helly's theorem on convex sets. For general curves, let us assume that $|S'| > 4$ and $\mathrm{VR}(p, S') = \emptyset$ hold. Let $q_1, q_2, q_3 \in S' \setminus \{p\}$ be pairwise different. By induction, there exist points $x_i \in \mathrm{VR}(p, S' \setminus \{q_i\})$. Since no x_i lies in $\mathrm{VR}(p, S')$ we have $x_i \in D(q_i, p)$. By (A1), there are paths P_{ij} connecting x_i and x_j in $\mathrm{VR}(p, S' \setminus \{q_i, q_j\}) \subseteq D(p, q_k)$, where $\{i, j, k\} = \{1, 2, 3\}$. Furthermore, P_{ij} has to be contained in $D(q_i, p) \cup D(q_j, p)$, or $\mathrm{VR}(p, S')$ would not be empty. Since $J(q_i, p)$ has to separate x_i from x_j and x_k, it must intersect P_{ij} and P_{ik} but not P_{jk}, because P_{jk} is contained in $D(p, q_i)$. Thus, $J(p, q_1)$ and $J(p, q_2)$ must intersect transversally (by (A5)) in a point w contained in a domain D bounded by the paths P_{12}, P_{23} and P_{13}; see Figure 4.

The same holds for the other bisector pairs. Let $J(p, q_i)$ be oriented such that $D(p, q_i)$ lies to the right. Then $J(p, q_3)$ has to intersect the part of $J(p, q_2)$ before w and the part of $J(p, q_1)$ after w, or the other way round, or $J(p, q_3)$ cuts through w between the parts of $J(p, q_1)$ and $J(p, q_2)$ before w; see Figure 5.

In the first case a face of the farthest region $\mathrm{VR}^*(p, \{p, q_1, q_2, q_3\})$ would be bounded, contradicting Lemma 7.

In the second case there would be a nonempty part of $\mathrm{VR}(p, \{p, q_1, q_2, q_3\})$ contained in D. Let x be a point of this subset. Because of $\mathrm{VR}(p, S') = \emptyset$ there must be $q \in S \setminus \{p, q_1, q_2, q_3\}$ such that $x \in D(q, p)$. But all paths P_{ij} are contained in $D(p, q)$, implying that $J(p, q)$ is closed—a contradiction to (A4).

In the third case, since $\mathrm{VR}(p, \{p, q_1, q_2, q_3\})$ must not be empty, by (A3), $J(p, q_3)$ has to intersect $J(p, q_1)$ or $J(p, q_2)$ in another point, resulting in a disconnection of $\mathrm{VR}(p, \{p, q_1, q_3\})$ or $\mathrm{VR}(p, \{p, q_2, q_3\})$ that contradicts (A1).

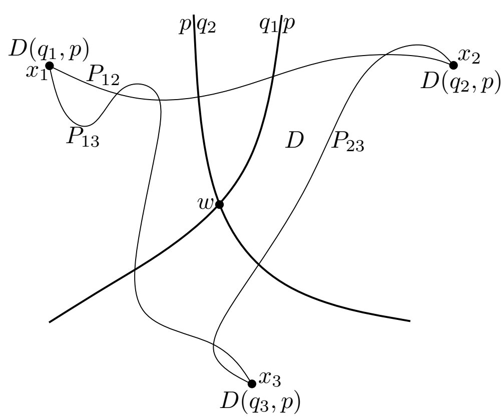

Fig. 4: In the proof of Lemma 1, curves $J(p, q_1)$ and $J(p, q_2)$ meet at point w.

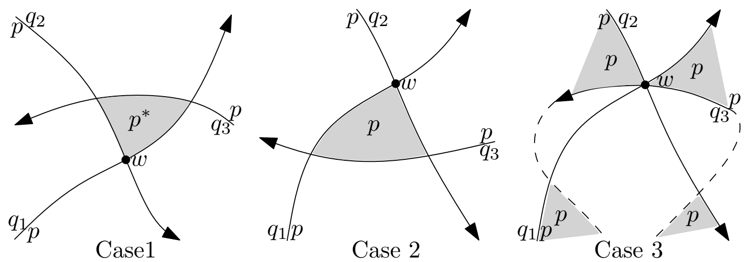

Fig. 5: Discussion of three cases.

Proof. (**of Lemma 7**) Suppose some farthest region $V^*(p, S)$ has a face F that is bounded, and let $J(p, q_1), \ldots, J(p, q_i)$ be bounding F in this order. Note that the indices q_j need not be pairwise different, but consecutive edges belong to different bisecting curves. Let x_j be the intersection point between $J(p, q_j)$ and $J(p, q_{j+1})$ on the boundary of F and let the bisectors $J(p, q_j)$ be such oriented that $D(p, q_j)$ lies on their left side.
If $i = 1$, then the bisector $J(p, q_1)$ would have to be closed, a contradiction.
Now let $i > 1$. By induction, if q_i is removed, the remaining bisectors $J(p, q_1)$, $\ldots, J(p, q_{i-1})$ do not bound a bounded face of $\mathrm{VR}^*(p, \{p, q_1, \ldots q_{i-1}\})$. Hence there is a part of $J(p, q_i)$ such that, w.l.o.g., the parts of $J(p, q_1)$ before x_1, and of $J(p, q_{i-1})$ after x_{i-1}, do not intersect such that $\mathrm{VR}^*(p, \{p, q_1, \ldots q_{i-1}\})$ gets bounded. By axiom (A3), the region $\mathrm{VR}(p, \{p, q_1, q_{i-1}\})$ is not empty. Thus the part of $J(p, q_1)$ after x_i must cross the part of $J(p, q_{i-1})$ before x_{i-2} at some point z; see Figure 6. Since $\mathrm{VR}(p, \{p, q_1, q_{i-1}, q_i\})$ is not empty, the part of $J(p, q_i)$ before x_{i-1} has to intersect $J(p, q_1)$, or the part of $J(p, q_i)$ after x_1 must intersect $J(p, q_{i-1})$, as shown in Figure 6. But in the former case, $\mathrm{VR}(p, \{p, q_1, q_i\})$ would be disconnected, in the latter case, $\mathrm{VR}(p, \{p, q_{i-1}, q_i\})$, contradicting axiom (A1). Here we are using axiom (A5) to ensure that, e. g., $J(p, q_i)$ and $J(p, q_{i-1})$ intersect transversally at w, so that there must be a non-empty wedge of $\mathrm{VR}(p, \{p, q_{i-1}, q_i\})$ at w.

It remains to show that $V^*(S)$ is connected. Suppose there is a curve L separating parts of $V^*(S)$. Then $L \subset \mathrm{VR}^*(p, S)$ for a $p \in S$, $L \cap D(p, q) = \emptyset$ for all $q \in S \setminus \{p\}$ and there are $q \neq r \in S$ such that $D(p, q)$ lies on one side of L and $D(p, r)$ on the other side. But then $\mathrm{VR}(p, \{p, q, r\})$ would be empty.

Proof. (**of Lemma 10**) First we show that for $|S| = 3$ each $P(S)$ fulfilling the above properties can be realized by an AVD. Then we assume $|S| \geq 3$ and consider $V(S)$ such that each triple p, q, r of sites change their ordering on Γ according to $P(S)$. This is possible because if the curve system of each triple of sites is admissible then the curve system of S is admissible, too; see [18]. Now if there are two bisectors $J(p, q)$ and $J(r, t)$ having a different order on Γ than p, q, r, t have in $P(S)$, then p, q, r, t are pairwise different, and neither of the bisectors $J(p, r), J(p, t), J(q, r)$ or $J(q, t)$ can occur between the two bisectors $J(p, q)$ and $J(r, t)$ on Γ. Otherwise, suppose w. l. o. g. that $J(p, r)$ occurs between

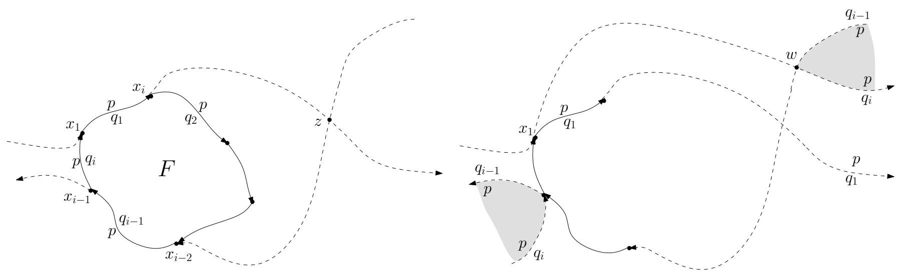

Fig. 6: Farthest AVDs cannot contain bounded regions.

$J(p,q)$ and $J(r,t)$; then $\{p,q,r\}$ would not have the same ordering on Γ as in $P(S)$, a contradiction to our assumption. Thus the ordering of $J(p,q)$ and $J(r,t)$ on Γ can be changed without changing the structure of $V(S)$.
Now let $S=\{p,q,r\}$. Then there are three different cases:

(1) Each site switches into the first position exactly once.
(2) One site switches into the first position exactly twice; it cannot do so more often because then it would have to switch with one of the other sites more often than twice. Further it implies that all other sites must switch themselves into first position exactly once.
(3) One site never moves to first position. This implies that both the other sites switch to first position exactly once; otherwise, either one site would remain in first position during the whole permutation, but then it would never switch with any other site, or the two other sites would have to switch more than twice.

Let $P_0=(p,q,r)$. Then there are two possibilities for P_1 in case (1): Either $P_1=(q,p,r)$, which leads to the sequence
$P_0=(p,q,r)$
$P_1=(q,p,r)$
$P_2=(q,r,p)$, otherwise r never switches into first position or p switches into first position twice
$P_3=(r,q,p)$, otherwise r never switches into first position
$P_4=(r,p,q)$, otherwise q switches into first position a second time
$P_5=(p,r,q)$, otherwise p and q switch more than twice
$P_6=(p,q,r)$, otherwise $P_0\neq P_6$;
or $P_1=(p,r,q)$, which leads to the same permutation sequence in inverse order.

Assume that p is the site that switches into first position twice in case (2). Then we get the following permutation sequence:
$P_0=(p,q,r)$
$P_1=(q,p,r)$, then
$P_2=(p,q,r)$, otherwise $P_2=(q,r,p)$ which leads to the permutation sequence as in case (1)

$P_3 = (p, r, q)$, otherwise p and q switch more than twice
$P_4 = (r, p, q)$, otherwise r never switches into first position
$P_5 = (p, r, q)$, otherwise p and q switch more than twice
$P_6 = (p, q, r)$, otherwise $P_0 \neq P_6$;
or in inverse order.
Assume that r is the site that never switches into first position in case (3). Then we get the following permutation sequence:
$P_0 = (p, q, r)$
$P_1 = (q, p, r)$, then
$P_2 = (q, r, p)$, otherwise p switches into first position twice
$P_3 = (q, p, r)$, otherwise r switches into first position
$P_4 = (p, q, r)$, otherwise p and r switch more than twice
$P_5 = (p, r, q)$, otherwise p and q switch more than twice
$P_6 = (p, q, r)$, otherwise $P_0 \neq P_6$;
or in inverse order.
These permutation sequences can be realized by the AVDs depicted in Figure 7.

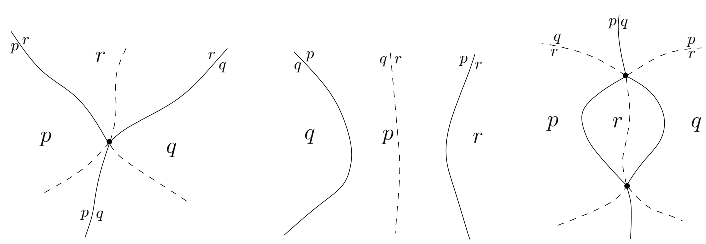

Fig. 7: Illustrations of cases (1) to (3) in the proof of Lemma 10.

Proof. (**of Lemma 15**) Consider $V^k(S) \cup \Gamma$, cut off all edges outside of Γ, and let G be the resulting graph. Then G is a connected planar graph and for its number of faces, f, of vertices, v, and edges, e, we have $f = F_k + 1$, $v = V_k + S_k$, $e = E_k + S_k$. Because of the general position assumption each vertex is of degree 3 and hence $2e = 3v$. Now the Euler formula $v - e + f = c + 1$ implies the lemma.

Proof. (**of Lemma 16**) Let V_k, V_k' and V_k'' be the number of Voronoi vertices, *new* Voronoi vertices and *old* Voronoi vertices in $V^k(S)$, respectively. Then because of Lemma 6 we have $V_k = V_k' + V_k'' = V_k' + V_{k-1}'$.
Claim 1: $F_{k+2} = E_{k+1} - 2V_k'$.
Because of Lemma 6 every *old* vertex of $V^{k+1}(S)$ lies in the interior of a face of $V^{k+2}(S)$. Consider a face F_i of $V^{k+2}(S)$. Let m_i be the number of *old* vertices of $V^{k+1}(S)$ enclosed in its interior. Then F_i encloses $e_i = 2m_i + 1$ edges of $V^{k+1}(S)$;

see Lemma 14. If we sum up through all the faces in $V^{k+2}(S)$, we obtain

$$\sum_{i=1}^{F_{k+2}} e_i = 2 \sum_{i=1}^{F_{k+2}} m_i + F_{k+2}.$$

Note that $\sum_{j=1}^{F_{k+2}} m_j = V''_{k+1} = V'_k$ and $\sum_{j=1}^{F_{k+2}} e_j = E_{k+1}$, hence $F_{k+2} = E_{k+1} - 2V'_k$.

Claim 2: The number of faces in $V^1(S)$ is $F_1 = n$ and the number of faces in $V^2(S)$ is $F_2 = 3(n-1) - S_1$.

The first part follows from axioms (A1) and (A3). To prove the second part, consider a face of $V^2(S)$. There are no *old* vertices in $V^1(S)$, therefore the face encloses exactly one edge of $V^1(S)$ and hence $F_2 = E_1$. Equation (1) implies $F_2 = 3(n-1) - S_1$.

Now we sum up F_{k+2} and F_{k+3} to obtain $F_{k+3} = E_{k+2} + E_{k+1} - F_{k+2} - 2V'_{k+1} - 2V'_k = E_{k+2} + E_{k+1} - F_{k+2} - 2V_{k+1}$; see Claim 1. Substituting (1) and (2) into it results in $F_{k+3} = 2F_{k+2} - F_{k+1} - 2 - S_{k+2} + S_{k+1}$. Using the iterative formula, the base cases $F_1 = n$ and $F_2 = 3(n-1) - S_1$, we derive the lemma by strong induction.