

Graph unique-maximum and conflict-free colorings

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Abstract

We investigate the relationship between two kinds of vertex colorings of graphs: unique-maximum colorings and conflict-free colorings. In a unique-maximum coloring, the colors are ordered, and in every path of the graph the maximum color appears only once. In a conflict-free coloring, in every path of the graph there is a color that appears only once. We also study computational complexity aspects of conflict-free colorings and prove a completeness result. Finally, we improve lower bounds for those chromatic numbers of the grid graph.

Keywords: unique-maximum coloring, ordered coloring, vertex ranking, conflict-free coloring

1 Introduction

In this paper we study two types of vertex colorings of graphs, both related to paths. The first one is the following:

Definition 1.1. A *unique-maximum coloring with respect to paths* of $G = (V, E)$ with k colors is a function $C: V \rightarrow \{1, \dots, k\}$ such that for each path p in G the maximum color occurs exactly once on the vertices of p . The minimum k for which a graph G has a unique-maximum coloring with k colors is called the *unique-maximum chromatic number* of G and is denoted by $\chi_{\text{um}}(G)$.

Unique maximum colorings are known alternatively in the literature as *ordered colorings* or *vertex rankings*. The problem of computing unique-maximum colorings is a well-known and widely studied problem (see e.g. [13]) with many applications including VLSI design [15] and parallel Cholesky factorization of matrices [16]. The problem is also interesting for the Operations Research community, because it has applications in planning efficient assembly of products in manufacturing systems [12]. In general, it seems that the vertex ranking problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.)

The other type of vertex coloring can be seen as a relaxation of the unique-maximum coloring.

Definition 1.2. A *conflict-free coloring with respect to paths* of $G = (V, E)$ with k colors is a function $C: V \rightarrow \{1, \dots, k\}$ such that for each path p in G there is a color that occurs exactly once on the vertices of p . The minimum k for which a graph G has a conflict-free coloring with k colors is called the *conflict-free chromatic number* of G and is denoted by $\chi_{\text{cf}}(G)$.

Conflict-free coloring of graphs with respect to paths is a special case of conflict-free colorings of hypergraphs, studied in Even et al. [9] and Smorodinsky [21]. One of the applications of conflict-free colorings is that it represents a frequency assignment for cellular networks. A cellular network

consists of two kinds of nodes: *base stations* and *mobile agents*. Base stations have fixed positions and provide the backbone of the network; they are represented by vertices in V . Mobile agents are the clients of the network and they are served by base stations. This is done as follows: Every base station has a fixed frequency; this is represented by the coloring C , i.e., colors represent frequencies. If an agent wants to establish a link with a base station it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not used by some other base station in the range. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, where possible, is preferable. Conflict-free coloring problems have been the subject of many recent papers due to their practical and theoretical interest (see e.g. [18, 11, 6, 8, 3]). Most approaches in the conflict-free coloring literature use unique-maximum colorings (a notable exception is the 'triples' algorithm in [3]), because unique-maximum colorings are easier to argue about in proofs, due to their additional structure. Another advantage of unique-maximum colorings is the simplicity of computing the unique color in any range (it is always the maximum color), given a unique-maximum coloring, which can be helpful if very simple mobile devices are used by the agents.

For general graphs, finding the exact unique-maximum chromatic number of a graph is NP-complete [20, 17] and there is a polynomial time $O(\log^2 n)$ approximation algorithm [5], where n is the number of vertices. Since the problem is hard in general, it makes sense to study specific graphs.

The $m \times m$ grid, G_m , is the *cartesian product* of two paths, each of length $m - 1$, that is, the vertex set of G_m is $\{0, \dots, m - 1\} \times \{0, \dots, m - 1\}$ and the edges are $\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| \leq 1\}$. It is known [13] that for general planar graphs the unique-maximum chromatic number is $O(\sqrt{n})$. Grid graphs are planar and therefore the $O(\sqrt{n})$ bound applies. One might expect that, since the grid has a relatively simple and regular structure, it should not be hard to calculate its unique-maximum chromatic number. This is why it is rather striking that, even though it is not hard to show upper and lower bounds that are only a small constant multiplicative factor apart, the *exact* value of these chromatic numbers is not known, and has been the subject of [1, 2].

Paper organization. In the rest of this section we provide the necessary definitions and some earlier results. In section 2, we prove that it is coNP-complete to decide whether a given vertex coloring of a graph is conflict-free with respect to paths. In section 3, we show that for every graph $\chi_{\text{um}}(G) \leq 2^{\chi_{\text{cf}}(G)} - 1$ and provide a sequence of graphs for which the ratio $\chi_{\text{um}}(G)/\chi_{\text{cf}}(G)$ tends to 2. In section 4, we introduce two games on graphs that help us relate the two chromatic numbers for the square grid graph. In section 5, we show a lower bound on the unique-maximum chromatic number of the square grid graph, improving previous results. Conclusions and open problems are presented in section 6.

1.1 Preliminaries

Note that definition 1.1 is not the typical definition found in the literature. Instead the more standard definition is the following.

Definition 1.3. A *unique-maximum k -coloring* (with respect to paths) of a graph G is a function $C: V(G) \rightarrow \{1, \dots, k\}$ such that for every pair of distinct vertices v, v' , and every path p from v to v' , if $C(v) = C(v')$, there is an internal vertex v'' of p such that $C(v) < C(v'')$.

It is not hard to show that the two definitions are equivalent (see, for example, [13]).

Definition 1.4. A graph X is a *minor* of Y , denoted as $X \preceq Y$, if X can be obtained from Y by a sequence of the following three operations: vertex deletion, edge deletion, and edge contraction. Edge contraction is the process of merging both endpoints of an edge into a new vertex, which is connected to all vertices adjacent to the two endpoints.

It is not difficult to prove, with the help of a recoloring argument, that the unique maximum chromatic number is monotone with respect to minors (see for example [4], lemma 4.3). We reproduce a proof here.

Proposition 1.5. *If $X \preceq Y$, then $\chi_{\text{um}}(X) \leq \chi_{\text{um}}(Y)$.*

Proof. Given a unique-maximum coloring C of G , if G' results from G after one of the three graph minor operations, then there is a unique-maximum coloring C' of G' with at most the number of colors of C . For the vertex and edge deletion operations, just set C' to be the restriction of C on G . Then, C' is unique-maximum, because G' contains a subset of the simple paths of G . For the edge contraction operation, say along edge $\{x, y\}$, which gives rise to the new vertex v_{xy} , set $C'(v_{xy}) = \max(C(x), C(y))$, and for every other vertex v of G' , set $C'(v) = C(v)$. If a path of G' does not contain v_{xy} , then it has the unique-maximum property, because the same path with the same coloring exists in G . If a path p' of G' contains v_{xy} , then there is a path p in G with the same vertex set as p' , except of course v_{xy} , which is replaced by at least one of x, y in p . Because of the choice of color of v_{xy} in C' , if p has the unique-maximum property then also p' has the unique-maximum property. Therefore, C' is unique-maximum. \square

The above proof implies a specific process to transform a coloring of a graph G when applying one of the graph minor operations, so that the coloring remains unique-maximum for the resulting graph: if you delete a vertex or an edge, leave the coloring as it is in the remaining vertices, and if you contract an edge give to the resulting vertex the maximum color of the two endpoints of the edge and leave the rest of the coloring as it is in the remaining vertices. Here is a lemma that will be useful in proving lower bounds on the unique-maximum chromatic number.

Lemma 1.6. *Assume C is an optimal unique-maximum coloring of G and after a sequence of graph minor operations you get coloring C' of G' . If C' is using x colors less than C , then $\chi_{\text{um}}(G) \geq \chi_{\text{um}}(G') + x$.*

Proof. Since C' is a unique-maximum coloring of G' , we have $\chi_{\text{um}}(G') \leq \chi_{\text{um}}(G) - x$. \square

The (traditional) chromatic number of a graph is denoted by $\chi(G)$ and is the smallest number of colors in a vertex coloring for which adjacent vertices are assigned different colors. A simple relation between the chromatic numbers we have defined so far is the following.

Proposition 1.7. *For every graph G , $\chi(G) \leq \chi_{\text{cf}}(G) \leq \chi_{\text{um}}(G)$.*

Proof. Since every unique-maximum coloring is also a conflict-free coloring, we have $\chi_{\text{cf}}(G) \leq \chi_{\text{um}}(G)$. A traditional coloring can be defined as a coloring in which paths of length one are conflict-free. Therefore every conflict-free coloring is also a traditional coloring and thus $\chi(G) \leq \chi_{\text{cf}}(G)$. \square

Moreover, we prove that both conflict-free and unique-maximum chromatic numbers are monotone under taking subgraphs.

Proposition 1.8. *If $X \subseteq Y$, then $\chi_{\text{cf}}(X) \leq \chi_{\text{cf}}(Y)$ and $\chi_{\text{um}}(X) \leq \chi_{\text{um}}(Y)$.*

Proof. Take the restriction of any conflict-free or unique-maximum coloring of graph Y to the vertex set $V(X)$. This is a conflict-free or unique maximum coloring of graph X , respectively, because the set of paths of graph X is a subset of all paths of Y . \square

If v is a vertex (resp. S is a set of vertices) of graph $G = (V, E)$, denote by $G - v$ (resp. $G - S$) the graph obtained from G by deleting vertex v (resp. vertices of S) and adjacent edges.

Definition 1.9. A subset $S \subseteq V$ is a *separator* of a connected graph $G = (V, E)$ if $G - S$ is disconnected or empty. A separator S is *inclusion minimal* if no proper subset $S' \subset S$ is a separator.

2 Deciding whether a coloring is conflict-free

In this section, we show a difference between the two chromatic numbers χ_{um} and χ_{cf} , from the computational complexity aspect. For the notions of complexity classes, hardness, and completeness, we refer, for example, to [19].

As we mentioned before, in [20, 17], it is shown that computing χ_{um} for general graphs is NP-complete. To be exact the following problem is NP-complete: “Given a graph G and an integer k , is it true that $\chi_{\text{um}}(G) \leq k$?”. The above fact implies that it is possible to check in polynomial time whether a given coloring of a graph is unique-maximum with respect to paths. We remark that both the conflict-free and the unique-maximum properties have to be true in every path of the graph. However, a graph with n vertices can have exponential in n number of distinct sets of vertices, each one of which is a vertex set of a simple path in the graph. For unique-maximum colorings we can find a shortcut as follows: Given a (connected) graph G and a vertex coloring of it, consider the set of vertices S of unique colors. Let $u, v \in V \setminus S$ such that they both have the maximum color that appears in $V \setminus S$. If there is a path in $G - S$ from u to v , then this path violates the unique maximum property. Therefore, S has to be a separator in G , which can be checked in polynomial time, otherwise the coloring is not unique-maximum. If $G - S$ is not empty, we can proceed analogously for each of its components. For conflict-free colorings there is no such shortcut, unless $\text{coNP} = \text{P}$, as the following theorem implies.

Theorem 2.1. *It is coNP-complete to decide whether a given graph and a vertex coloring of it is conflict-free with respect to paths.*

Proof. In order to prove that the problem is coNP-complete, we prove that it is coNP-hard and also that it belongs to coNP.

We show coNP-hardness by a reduction from the complement of the Hamiltonian path problem. For every graph G , we construct in polynomial time a graph G^* of polynomial size together with a coloring C of its vertices such that G has no Hamiltonian path if and only if C is conflict-free with respect to paths of G^* .

Assume the vertices of graph G are v_1, v_2, \dots, v_n . Then, graph G^* consists of two isomorphic copies of G , denoted by \hat{G} and \check{G} , with vertex sets $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ and $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$, respectively. Additionally, for every $1 \leq i \leq n$, G^* contains the path

$$P_i = \bar{v}_i, v_{i,1}, v_{i,2}, \dots, v_{i,i-1}, v_{i,i+1}, \dots, v_{i,n}, \underline{v}_i,$$

where, for every i , $v_{i,1}, v_{i,2}, \dots, v_{i,i-1}, v_{i,i+1}, \dots, v_{i,n}$ are new vertices. We use the following notation for the two possible directions to traverse this path:

$$P_i^\downarrow = (v_{i,1}, \dots, v_{i,i-1}, v_{i,i+1}, \dots, v_{i,n}),$$

$$P_i^\uparrow = (v_{i,n}, \dots, v_{i,i+1}, v_{i,i-1}, \dots, v_{i,1}).$$

We call paths P_i *connecting paths*.

We now describe the coloring of $V(G^*)$. For every i , we set $C(\bar{v}_i) = C(\underline{v}_i) = i$. For every $i > j$, we set $C(v_{i,j}) = C(v_{j,i}) = n + \binom{i-1}{2} + j$. Observe that every color occurs exactly in two vertices of G^* .

If G has a Hamiltonian path, say $v_1 v_2 \dots v_n$, then there is a path through all vertices of G^* , either

$$\bar{v}_1 P_1^\downarrow \underline{v}_1 \underline{v}_2 P_2^\uparrow \bar{v}_2 \dots \bar{v}_{n-1} P_{n-1}^\downarrow \underline{v}_{n-1} \underline{v}_n P_n^\uparrow \bar{v}_n, \text{ if } n \text{ is even,}$$

or

$$\bar{v}_1 P_1^\downarrow \underline{v}_1 \underline{v}_2 P_2^\uparrow \bar{v}_2 \dots \underline{v}_{n-1} P_{n-1}^\uparrow \bar{v}_{n-1} \bar{v}_n P_n^\downarrow \underline{v}_n, \text{ if } n \text{ is odd.}$$

But then, this path has no uniquely occurring color and thus C is not conflict-free.

Suppose now that C is not a conflict-free coloring. We prove that G has a Hamiltonian path.

By the assumption, there is a path P in G^* which is not conflict-free. This path must contain none or both vertices of each color. Therefore, P can not be completely contained in \hat{G} , or in \check{G} , or in some P_i . Also, P can not contain only one of \bar{v}_i and \underline{v}_i , for some i . Therefore, P must contain both \bar{v}_i and \underline{v}_i for a non-empty subset of indices i .

Then, it must contain completely some P_i , because vertices in \hat{G} and \check{G} can only be connected with some complete P_i . But since each one of the $n - 1$ colors of this P_i occurs in a different connecting paths, P must contain a vertex in every connecting path. But then P must contain every \bar{v}_i and \underline{v}_i , because vertices in P_i can only be connected to the rest of the graph through one of \bar{v}_i or \underline{v}_i .

Suppose that P is not a Hamiltonian path of G^* . Observe that if P does not contain all vertices of some connecting path P_i , then one of its end vertices should be there. If P does not contain vertex $v_{i,j}$, then it can not contain $v_{j,i}$ either. But then one end vertex of P should be on P_i , the other one on P_j , and all other vertices of G^* are on P . Therefore, we can extend P such that it contains $v_{i,j}$ and $v_{j,i}$ as well. So assume in the sequel that P is a Hamiltonian path of G^* .

Now we modify P , if necessary, so that both of its end-vertices e and f lie in $V(\hat{G}) \cup V(\check{G})$. If e and f are adjacent in G^* , then add the edge ef to P and we get a Hamiltonian cycle of G^* . Now remove one of its edges which is either in \hat{G} , or in \check{G} and get the desired Hamiltonian path. Suppose now that e and f are not adjacent, and e is on one of the connecting paths. Then e should be adjacent to the end vertex e' of that connecting path, which is in \hat{G} or in \check{G} . Add edge ee' to P . We get a cycle and a path joined in e' . Remove the other edge of the cycle adjacent to e' . We have a Hamiltonian path now, whose end vertex is e' instead of e . Proceed analogously for f , if necessary.

Now we have a Hamiltonian path P of G^* with end-vertices in $V(\hat{G}) \cup V(\check{G})$. Then, P is in the form, say,

$$\bar{v}_1 P_1^\downarrow \underline{v}_1 \underline{v}_2 P_2^\uparrow \bar{v}_2 \dots \bar{v}_{n-1} P_{n-1}^\downarrow \underline{v}_{n-1} \underline{v}_n P_n^\uparrow \bar{v}_n, \text{ if } n \text{ is even,}$$

or

$$\bar{v}_1 P_1^\downarrow \underline{v}_1 \underline{v}_2 P_2^\uparrow \bar{v}_2 \dots \underline{v}_{n-1} P_{n-1}^\uparrow \bar{v}_{n-1} \bar{v}_n P_n^\downarrow \underline{v}_n, \text{ if } n \text{ is odd.}$$

But then, $v_1 v_2 \dots v_n$ is a Hamiltonian path in G .

Finally, the problem is in coNP because one can verify that a coloring of a given graph is not conflict-free in polynomial time, by giving the corresponding path. \square

We also show an example graph G , its transformation graph G^* , and its coloring C in figure 1.

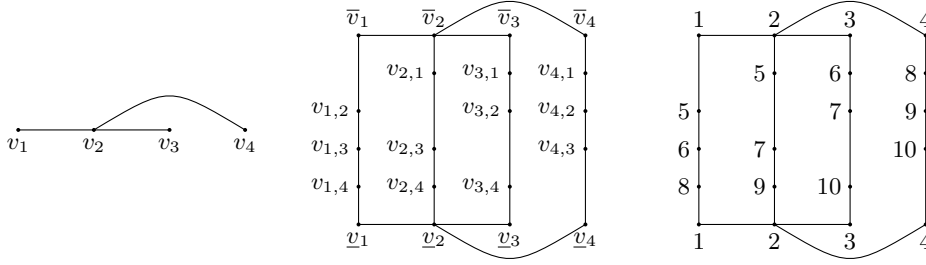


Figure 1: Example graph G , its transformation G^* , and coloring C of G^*

3 The two chromatic numbers of general graphs

We have seen that $\chi_{\text{um}}(G) \geq \chi_{\text{cf}}(G)$ (proposition 1.7). In this section we show that $\chi_{\text{um}}(G)$ can not be larger than an exponential function of $\chi_{\text{cf}}(G)$. We also provide an infinite sequence of graphs H_1, H_2, \dots , for which $\lim_{k \rightarrow \infty} (\chi_{\text{um}}(H_k) / \chi_{\text{cf}}(H_k)) = 2$.

The path of n vertices is denoted by P_n . It is known that $\chi_{\text{um}}(P_n) = \lfloor \log_2 n \rfloor + 1$ (see for example [9]).

Lemma 3.1. *For every path P_n , $\chi_{\text{cf}}(P_n) = \lfloor \log_2 n \rfloor + 1$.*

Proof. By proposition 1.7, $\chi_{\text{cf}}(P_n) \leq \chi_{\text{um}}(P_n)$. We prove a matching lower bound by induction. We have $\chi_{\text{cf}}(P_1) \geq 1$. For $n > 1$, there is a uniquely occurring color in any conflict-free coloring of the the whole path P_n . Then, $\chi_{\text{cf}}(P_n) \geq 1 + \chi_{\text{cf}}(P_{\lfloor n/2 \rfloor})$, which implies $\chi_{\text{cf}}(P_n) \geq \lfloor \log_2 n \rfloor + 1$. \square

Moreover, we are going to use the following result (lemma 5.1 of [13]): If the longest path of G has k vertices, then $\chi_{\text{um}}(G) \leq k$.

Proposition 3.2. *For every graph G , $\chi_{\text{um}}(G) \leq 2^{\chi_{\text{cf}}(G)} - 1$.*

Proof. Set $j = \chi_{\text{cf}}(G)$. Since $\chi_{\text{cf}}(G) \leq j$, for any path $P \subseteq G$, $\chi_{\text{cf}}(P) \leq j$, therefore, by lemma 3.1, the longest path has at most $2^j - 1$ vertices, so by lemma 5.1 of [13], $\chi_{\text{um}}(G) \leq 2^j - 1$. \square

We define recursively the following sequence of graphs: Graph H_0 is a single vertex. Suppose that we have already defined H_{k-1} . Then H_k consists of (a) a $K_{2^{k+1}-1}$, (b) $2^{k+1} - 1$ copies of H_{k-1} , and (c) for for each i , $1 \leq i \leq 2^{k+1} - 1$, the i -th vertex of the $K_{2^{k+1}-1}$ is connected by an edge to one of the vertices of the i -th copy of H_{k-1} .

Lemma 3.3. *For $k \geq 0$, $\chi_{\text{cf}}(H_k) = 2^{k+1} - 1$.*

Proof. By induction on k . For $k = 0$, $\chi_{\text{cf}}(H_0) = 1$. For $k > 0$, we have $H_k \supseteq K_{2^{k+1}-1}$, therefore, $\chi_{\text{cf}}(H_k) \geq 2^{k+1} - 1$.

In order to prove that $\chi_{\text{cf}}(H_k) \leq 2^{k+1} - 1$, it is enough to describe a conflict-free coloring of H_k with $2^{k+1} - 1$ colors, given a conflict-free coloring of H_{k-1} with $2^k - 1$ colors: We color the vertices of the clique $K_{2^{k+1}-1}$ with colors $1, 2, \dots, 2^{k+1} - 1$ such that the i -th vertex is colored with color i . Consider these colors mod $2^{k+1} - 1$, e. g. color 2^{k+1} is identical to color 1. Recall that the i -th copy of H_{k-1} has a vertex connected to the i -th vertex of $K_{2^{k+1}-1}$, and by induction we know that $\chi_{\text{cf}}(H_{k-1}) = 2^k - 1$. Color the i -th copy of H_{k-1} , with colors $i + 1, i + 2, \dots, i + 2^k - 1$.

We claim that this vertex coloring of H_k is conflict-free. If a path is completely contained in a copy of H_{k-1} , then it is conflict-free by induction. If a path is completely contained in the clique $K_{2^{k+1}-1}$, then it is also conflict-free, because all colors in the clique part are different. If a

path contains vertices from a single copy of H_{k-1} , say, the i -th copy, and the clique, then the i -th vertex of the clique is on the path and uniquely colored. The last case is when a path contains vertices from exactly two copies of H_{k-1} . Suppose that these are the i -th and j -th copies of H_{k-1} , $1 \leq i < j \leq 2^{k+1} - 1$. If $i + 2^k - 1 < j$, then color j is unique in the path; indeed, the i -th copy of H_{k-1} is colored with colors $i + 1, \dots, i + 2^k - 1$, and the j -th copy of H_{k-1} is colored with colors $j + 1, \dots, j + 2^k - 1$, while color j appears only once in $K_{2^{k+1}-1}$. Similarly, if $i + 2^k - 1 \geq j$, then color i is unique in the path. \square

Lemma 3.4. $\chi_{\text{um}}(H_k) \leq 2^{k+2} - k - 3$.

Proof. By induction. For $k = 0$, $\chi_{\text{um}}(H_0) = 1$. For $k > 0$, in order to color H_k use the $2^{k+1} - 1$ different highest colors for the clique part. By the inductive hypothesis $\chi_{\text{um}}(H_{k-1}) \leq 2^{k+1} - k - 2$. For each copy of H_{k-1} , use the same coloring with the $2^{k+1} - k - 2$ lowest colors. This coloring of H_k is unique maximum. Indeed, if a path is contained in a copy of H_{k-1} then it is unique maximum by induction, and if it contains a vertex in the clique part, then it is also unique maximum. The total number of colors is $2^{k+2} - k - 3$. \square

Lemma 3.5. *If Y is a graph that consists of a K_ℓ and ℓ isomorphic copies of a connected graph X , such that for $1 \leq i \leq \ell$ a vertex of it i -th copy is connected to the i -th vertex of K_ℓ by an edge. Then we have $\chi_{\text{um}}(Y) \geq \ell - 1 + \chi_{\text{um}}(X)$*

Proof. By induction on ℓ . For $\ell = 1$, we have that $\chi_{\text{um}}(Y) \geq \chi_{\text{um}}(X)$, because $Y \supseteq X$. For the inductive step, for $\ell > 1$, if Y consists of a K_ℓ and ℓ copies of X , then Y is connected, and thus contains a vertex v with unique color. But then, $Y - v \supseteq Y'$, where Y' is a graph that consists of a $K_{\ell-1}$ and $\ell - 1$ isomorphic copies of a X , each connected to a different vertex of $K_{\ell-1}$, and thus $\chi_{\text{um}}(Y) = 1 + \chi_{\text{um}}(Y') \geq \ell - 1 + \chi_{\text{um}}(X)$. \square

Lemma 3.6. $\chi_{\text{um}}(H_k) \geq 2^{k+2} - 2k - 3$.

Proof. By induction. For $k = 0$, $\chi_{\text{um}}(H_0) = 1$. For $k > 0$, by the inductive hypothesis and lemma 3.5, $\chi_{\text{um}}(H_k) \geq 2^{k+1} - 1 - 1 + 2^{k+1} - 2(k - 1) - 3 = 2^{k+2} - 2k - 3$. \square

Theorem 3.7. *We have $\lim_{k \rightarrow \infty} (\chi_{\text{um}}(H_k) / \chi_{\text{cf}}(H_k)) = 2$.*

Proof. From lemmas 3.3, 3.4, 3.6, we have

$$\frac{2^{k+2} - 2k - 3}{2^{k+1} - 1} \leq \frac{\chi_{\text{um}}(H_k)}{\chi_{\text{cf}}(H_k)} \leq \frac{2^{k+2} - k - 3}{2^{k+1} - 1}$$

which implies that the ratio tends to 2. \square

4 The two chromatic numbers of a square grid

In this section, we define two games on graphs, each played by two players. The first game characterizes completely the unique-maximum chromatic number of the graph. The second game is related to the conflict-free chromatic number of the graph. We use the two games to prove that the conflict-free chromatic number of the square grid is a function of the unique-maximum chromatic number of the square grid. This is useful because it allows to translate existing lower bounds on the unique-maximum chromatic number of the square grid to lower bounds on the corresponding conflict-free chromatic number.

The first game is the connected component game. It is played on a graph G by two players.

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 $i \leftarrow 0; G^0 \leftarrow G$ 
while  $V(G^i) \neq \emptyset$ :
  increment  $i$  by 1
  Player 1 chooses a connected component  $S^i$  of  $G^{i-1}$ 
  Player 2 chooses a vertex  $v_i \in S^i$ 
   $G^i \leftarrow G^{i-1}[S^i \setminus \{v_i\}]$ 

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The game is finite, because if G^i is not empty, then G^{i+1} is a strict subgraph of G^i . The result of the game is its length, that is, the final value of i . Player 1 tries to make the final value of i as large as possible and thus is the maximizer player. Player 2 tries to make the final value of i as small as possible and thus is the minimizer player. If both players play optimally, then the result is the *value* of the connected component game on graph G , which is denoted by $v^{\text{cs}}(G)$.

Proposition 4.1. *In the connected component game, there is a strategy for player 2 (the minimizer), so that the result of the game is at most $\chi_{\text{um}}(G)$, i.e., $v^{\text{cs}}(G) \leq \chi_{\text{um}}(G)$.*

Proof. By induction on $\chi_{\text{um}}(G)$: If $\chi_{\text{um}}(G) = 0$, i.e., the graph is empty, the value of the game is 0. If $\chi_{\text{um}}(G) = k > 0$, then in the first turn some connected component S^1 is chosen by player 1. Then, the strategy of player 2 is to take an optimal unique-maximum coloring C of G and choose a vertex v_1 in S^1 that has a unique color in S^1 . Then, $G^1 = G[S^1 \setminus \{v_1\}] \subset G^0$ and the restriction of C to $S^1 \setminus \{v_1\}$ is a unique-maximum coloring of G^1 that is using at most $k - 1$ colors. Thus, $\chi_{\text{um}}(G^1) \leq k - 1$, and by the inductive hypothesis player 2 has a strategy so that the result of the game on G^1 is at most $k - 1$. Therefore, player 2 has a strategy so that the result of the game on $G^0 = G$ is at most $1 + k - 1 = k$. \square

Lemma 4.2. *For every $v \in V(G)$, $\chi_{\text{um}}(G - v) \geq \chi_{\text{um}}(G) - 1$*

Proof. Assume for the sake of contradiction that there exists a $v \in V(G)$ for which $\chi_{\text{um}}(G - v) < \chi_{\text{um}}(G) - 1$. Then an optimal coloring of $G - v$ can be extended to a coloring of G , where v has a new unique maximum color. Therefore there is a coloring of G that uses less than $\chi_{\text{um}}(G) - 1 + 1 = \chi_{\text{um}}(G)$ colors; a contradiction. \square

Proposition 4.3. *In the connected component game, there is a strategy for player 1 (the maximizer), so that the result of the game is at least $\chi_{\text{um}}(G)$, i.e., $v^{\text{cs}}(G) \geq \chi_{\text{um}}(G)$.*

Proof. By induction on $\chi_{\text{um}}(G)$: If $\chi_{\text{um}}(G) = 0$, i.e., the graph is empty, the result of the game is zero. If $\chi_{\text{um}}(G) = k > 0$, the strategy of player 1 is to choose a connected component S^1 such that $\chi_{\text{um}}(G[S^1]) = k$. For every choice of v_1 by Player 2, by lemma 4.2, $\chi_{\text{um}}(G^1) \geq k - 1$, and thus, by the inductive hypothesis player 1 has a strategy so that the result of the game on G^1 is at least $k - 1$. Therefore, the result of the game on $G^0 = G$ is at least $1 + k - 1 = k$. \square

Corollary 4.4. *For every graph, $v^{\text{cs}}(G) = \chi_{\text{um}}(G)$.*

The second game is the path game.

```

 $i \leftarrow 0; G^0 \leftarrow G$ 
while  $V(G^i) \neq \emptyset$ :
  increment  $i$  by 1
  Player 1 chooses the set of vertices  $S^i$  of a path of  $G^{i-1}$ 
  Player 2 chooses a vertex  $v_i \in S^i$ 
   $G^i \leftarrow G^{i-1}[S^i \setminus \{v_i\}]$ 

```


The only difference with the connected component game is that in the path game the vertex set S^i that maximizer chooses is the vertex set of a path of the graph G^{i-1} . If both players play optimally, then the result is the *value* of the path game on graph G , which is denoted by $v^p(G)$.

Proposition 4.5. *In the path game, there is a strategy for player 2 (the minimizer), so that the result of the game is at most $\chi_{\text{cf}}(G)$, i.e., $v^p(G) \leq \chi_{\text{cf}}(G)$.*

Proof. By induction on $\chi_{\text{cf}}(G)$: If $\chi_{\text{cf}}(G) = 0$, i.e., the graph is empty, the value of the game is 0. If $\chi_{\text{cf}}(G) = k > 0$, then in the first turn some vertex set S^1 of a path of G is chosen by player 1. Then, the strategy of player 2 is to find an optimal conflict-free coloring C of G and choose a vertex v_1 in S^1 that has a unique color in S^1 . Then, $G^1 = G[S^1 \setminus \{v_1\}] \subset G^0$ and the restriction of C to $S^1 \setminus \{v_1\}$ is a conflict-free coloring of G^1 that is using at most $k - 1$ colors. Thus, $\chi_{\text{cf}}(G^1) \leq k - 1$, and by the inductive hypothesis player 2 has a strategy so that the result of the game is at most $k - 1$. Therefore, player 2 has a strategy so that the result of the game is at most $1 + k - 1 = k$. \square

A proposition analogous to 4.3 for the path game is not true. For example, for the complete binary tree of four levels (with 15 vertices, 8 of which are leaves), B_4 , it is not difficult to check that $v^p(B_4) = v^p(P_7) = 3$, but $\chi_{\text{cf}}(B_4) = 4$.

Now, we are going to concentrate on the square grid graph. Assume that m is even. We intend to translate a strategy of player 1 (the maximizer) on the connected component game for graph $G_{m/2}$ to a strategy for player on the path game for graph G_m .

Observe that for every connected graph G , there is an ordering of its vertices, v_1, v_2, \dots, v_n such that the subgraph induced by the first k vertices (for every $1 \leq k \leq n$) is also connected. Just pick a vertex to be v_1 , and add the other vertices one by one such that the new vertex v_i is connected to the graph induced by v_1, \dots, v_{i-1} . This is possible, since G itself is connected. We call such an ordering of the vertices an *always-connected ordering*.

Now we decompose the vertex set of G_m into groups of four vertices,

$$Q_{x,y} = \{(2x, 2y), (2x + 1, 2y), (2x, 2y + 1), (2x + 1, 2y + 1)\},$$

for $0 \leq x, y < m/2$, called *special quadruples*, or briefly quadruples. Let $W_m = \{Q_{x,y} \mid 0 \leq x, y < m/2\}$ and let $\tau(x, y) = Q_{x,y}$, a bijection between vertices of $V(G_{m/2})$ and W_m . Extend τ for subsets of vertices of $G_{m/2}$ in a natural way, for any $S \subseteq V(G_{m/2})$, $\tau(S) = \bigcup_{(x,y) \in S} \tau(x, y)$. Define also a kind of inverse τ' of τ as $\tau'(x, y) = (\lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ for any $0 \leq x, y < m$, and for any $S \subseteq V(G_m)$, $\tau'(S) = \{\tau'(x, y) \mid (x, y) \in S\}$.

Let $(x, y) \in V(G_{m/2})$. We call vertices $(x, y + 1)$, $(x, y - 1)$, $(x - 1, y)$, and $(x + 1, y)$, if they exist, the *upper*, *lower*, *left*, and *right neighbors* of (x, y) , respectively. Similarly, quadruples $Q_{x,y+1}$, $Q_{x,y-1}$, $Q_{x-1,y}$, and $Q_{x+1,y}$ the *upper*, *lower*, *left*, and *right neighbors* of $Q_{x,y}$, respectively.

Quadruple $Q_{x,y}$ induces four edges in G_m , $\{(2x+1, 2y), (2x+1, 2y+1)\}$, $\{(2x, 2y), (2x, 2y+1)\}$, $\{(2x, 2y), (2x, 2y+1)\}$, $\{(2x+1, 2y), (2x+1, 2y+1)\}$, we call them *upper*, *lower*, *left*, and *right edges* of $Q_{x,y}$.

By *direction* d , we mean one of the four basic directions, *up*, *down*, *left*, *right*. For a given set $S \subseteq V(G_{m/2})$, we say that $v \in S$ is *open* in S in direction d , if its neighbor in direction d is not in S . In this case we also say that $\tau(v)$ is open in $\tau(S)$ in direction d .

Lemma 4.6. *If S induces a connected subgraph in $G_{m/2}$, then there is a path in G_m whose vertex set is $\tau(S)$.*

Proof. We prove a stronger statement: If S induces a connected subgraph in $G_{m/2}$, then there is a cycle C in G_m whose vertex set is $\tau(S)$, and if $v \in S$ is open in direction d in S , then C contains the d -edge of $\tau(v)$.

The proof is by induction on $|S| = k$. For $k = 1$, $\tau(S)$ is one quadruple and we can take its four edges.

Suppose that the statement has been proved for $|S| < k$, and assume that $|S| = k$. Consider an always-connected ordering v_1, v_2, \dots, v_k of S . Let $S' = S \setminus v_k$. By the induction hypothesis, there is a cycle C' satisfying the requirements. Vertex v_k has at least one neighbor in S' , say, v_k is the neighbor of v_i in direction d . But then, v_i is open in direction d in S' , therefore, C' contains the d -edge of $\tau(v_i)$. Remove this edge from C' and substitute by a path of length 5, passing through all four vertices of $\tau(v_k)$. The resulting cycle, C , contains all vertices of $\tau(S)$, it contains each edge of $\tau(v_k)$, except the one in the opposite direction to d , and it contains all edges of C' , except the d -edge of $\tau(v_k)$, but v_k is not open in S in direction d . This concludes the induction step, and the proof. \square

Proposition 4.7. *For every $m > 1$, $v^p(G_m) \geq v^{cs}(G_{\lfloor m/2 \rfloor})$.*

Proof. Assume, without loss of generality that m is even (if not work with graph G_{m-1} instead). In order, to prove that $v^p(G_m) \geq v^{cs}(G_{\lfloor m/2 \rfloor})$ it is enough, given a strategy for player 1 in the connected set game for $G_{m/2}$, to construct a strategy for player 1 (the maximizer) in the path game for G_m , so that the result of the path game is at least as much as the result of the connected set game. We present the argument as if player 1, apart from the path game, plays in parallel a connected set game on $G_{m/2}$ (for which player 1 has a given strategy to choose connected sets in every round), where player 1 also chooses the moves of player 2 in the connected set game.

At round i of the path game on G_m , player 1 simulates round i of the connected set game on $G_{m/2}$. At the start of round i , player 1 has a graph $G^{i-1} \subseteq G_m$ in the path game and a graph $\hat{G}^{i-1} \subseteq G_{m/2}$ in the connected set game. Player 1 chooses a set \hat{S}^i in the simulated connected set game from his given strategy, and then constructs the path-spanned set $S^i = \tau(\hat{S}^i)$ (by lemma 4.6) and plays it in the path game. Then player 2 chooses a vertex $v_i \in S^i$. Player 1 computes $\hat{v}_i = \tau'(v_i)$ and simulates the move \hat{v}_i of player 2 in the connected set game. This is a legal move for player 2 in the connected set game because $\hat{v}_i \in \hat{S}^i$.

We just have to prove that $S^i = \tau(\hat{S}^i)$ is a legal move for player 1 in the path game, i.e., $S^i \subseteq V(G^{i-1})$. We also have to prove $S^i = \tau(\hat{S}^i)$ is spanned by a path in G^{i-1} but this is always true by lemma 4.6, since \hat{S}^i is a connected vertex set in \hat{G}^{i-1} . Since $S^i \subseteq \tau(V(\hat{G}^{i-1}))$, it is enough to prove that at round i , $\tau(V(\hat{G}^{i-1})) \subseteq V(G^{i-1})$. The proof is by induction on i . For $i = 1$, $G^0 = G_m$, $\hat{G}^0 = G_{m/2}$, and thus $\tau(V(\hat{G}^0)) = V(G^0)$. At the start of round i with $i > 1$, $\tau(V(\hat{G}^{i-1})) \subseteq V(G^{i-1})$, by the inductive hypothesis. Then, $\tau(\hat{S}^i) = S^i$ and $\tau(\hat{S}^i \setminus \{\hat{v}_i\}) = \tau(\hat{S}^i) \setminus \tau(\hat{v}_i) = S^i \setminus \tau(\hat{v}_i) \subseteq S^i \setminus \{v_i\}$, because $v_i \in \tau(\hat{v}_i)$. Thus, $\tau(V(\hat{G}^{i-1}[\hat{S}^i \setminus \{\hat{v}_i\}])) \subseteq V(G^{i-1}[S^i \setminus \{v_i\}])$, i.e., $\tau(V(\hat{G}^i)) \subseteq V(G^i)$. \square

Theorem 4.8. *For every $m > 1$, $\chi_{cf}(G_m) \geq \chi_{um}(G_{\lfloor m/2 \rfloor})$.*

Proof. By proposition 4.5, $\chi_{cf}(G_m) \geq v^p(G_m)$, by proposition 4.7, $v^p(G_m) \geq v^{cs}(G_{\lfloor m/2 \rfloor})$, and by proposition 4.3, $v^{cs}(G_{\lfloor m/2 \rfloor}) \geq \chi_{um}(G_{\lfloor m/2 \rfloor})$. \square

5 Lower bounds on the chromatic numbers of the square grid

In this section, we prove an asymptotic lower bound of $5m/3$ on the unique-maximum chromatic number of the $m \times m$ grid graph. In any unique-maximum coloring of a connected graph G the set of vertices U with uniquely occurring colors in G must be a separator. Some subset of U must be an inclusion minimal separator, i.e., in every unique-maximum coloring of a connected graph G , there

is an inclusion minimal separator with uniquely colored vertices. Our method for proving a lower bound on $\chi_{\text{um}}(G_m)$ will be the following. We will consider all possible optimal unique-maximum colorings of G_m . For each optimal coloring C , we will argue about the form of inclusion minimal separators with uniquely colored vertices in C . Then, using the three graph minor operations, we will get a coloring C' of some graph $G' \preceq G_m$ that is using x less colors than C . Then, by lemma 1.6, $\chi_{\text{um}}(G_m) \geq \chi_{\text{um}}(G') + x$. Finally, we will get a lower bound on $\chi_{\text{um}}(G_m)$ by showing that G' contains a large grid as a minor (or as a subgraph) and using induction.

We improve on the $3m/2$ lower bound on $\chi_{\text{um}}(G_m)$, for $m \geq 2$, given in [2]; we use a different case analysis to prove a lower bound of $5m/3 - \log_{\frac{5}{2}} m$, for $m \geq 2$.

Before proceeding to the proof of the lower bounds, we should state some auxiliary results, related to separators in general, and also to the form of separators in grid-like graphs. A similar analysis is provided in [2], but we also provide it here for completeness. We start with the following lemma, which is mentioned without proof in [14] and is similar to an exercise in [10].

Lemma 5.1. *A separator S of a connected graph G is inclusion minimal if and only if every vertex of S has a vertex adjacent in every connected component of $G - S$.*

Proof. If S is an inclusion minimal separator, then, for the sake of contradiction, assume there is a vertex $v \in S$ which has no vertex adjacent in some connected component C of $G - S$. Since v is not adjacent with any vertex of C , every path in G from a vertex of C to a vertex in the other connected components contains a vertex in $S \setminus \{v\}$. Therefore, $S \setminus \{v\}$ is still a separator, contradicting the inclusion minimality of S .

If every vertex $v \in S$ has a vertex adjacent in every connected component of $G - S$, then for every $v \in S$, $G - (S \setminus \{v\})$ consists of a single connected component, and thus $S \setminus \{v\}$ is not a separator, i.e., S is inclusion minimal. \square

We are now ready to state and prove some facts about the form of inclusion minimal separators in G_m . We remark that any inclusion minimal separator S of G_m for $m \geq 2$ has size $|S| < m^2$.

Lemma 5.2. *If S is an inclusion minimal separator of G_m , where $m \geq 2$, then $G_m - S$ consists of exactly two connected components.*

Proof. Assume for the sake of contradiction that $G_m - S$ consists of at least three connected components, say A , B and C . Without loss of generality, assume there is a vertex $v = (x, y)$ of the separator with adjacent vertices $a = (x - 1, y)$, $b = (x + 1, y)$, and $c = (x, y + 1)$, such that $a \in A$, $b \in B$, and $c \in C$ (see figure 2). Then, $u = (x - 1, y + 1)$ must be in the separator S , because of the

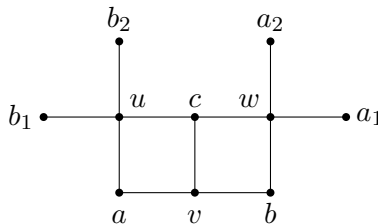


Figure 2: Impossibility of three connected components in $G_m - S$

path auc , and $w = (x + 1, y + 1)$ must also be in the separator S , because of the path bwc . Then, by lemma 5.1, u must be adjacent to a vertex b' in B (at least one of vertices b_1, b_2 in figure 2), and w must be adjacent to a vertex a' in A (at least one of vertices a_1, a_2 in figure 2).

Consider the embedding of G_m in the plane with the standard drawing, where the edges are straight line segments. Moreover, simple paths in G_m induce simple curves in the above embedding. Since a and a' are in the same connected component A , there is a simple path connecting a and a' contained completely in A . There is also the simple path from a to a' through v , c , and w , which does not intersect with the previous path. Those two paths together form a simple closed curve K_a in the embedding (a Jordan curve). By the Jordan curve theorem, the Jordan curve K_a divides the plane into two maximal connected subsets, that we call regions (we do not use the more standard term ‘connected components’ to avoid confusion with connected components of $G_m - S$), and each region’s boundary is exactly the Jordan curve K_a . Consider the following two subsets of the plane: s_1 is square $bvcw$ minus the segments vc and vw , and s_2 is square $avcu$ minus the segments av and vc . Sets s_1 and s_2 are contained in different regions, because part of their boundary (i.e., segment vc) is contained in K_a , and because K_a does not intersect neither s_1 (since $b \notin K_a$) nor s_2 (since $u \notin K_a$). Therefore, $b \in s_1$ and $u \in s_2$ are in different regions. Moreover, b' and u are in the same region, because curve K_a can not intersect the straight line segment $b'u$. Therefore, b and b' are in different regions. Since b and b' are in the same connected component B of $G_m - S$, there is a simple path from b to b' consisting only of vertices in B . This simple path induces a simple curve K_b in the embedding of G_m and, since b and b' are in different regions, this curve intersects the closed curve K_a at some point. Because these curves arise from paths in G_m , they can only intersect at some vertex point, which is a contradiction because curve K_a consists only of vertices in A , C , and S (not B). \square

We say that two vertices (x_1, y_1) , (x_2, y_2) of a separator of G_m are *neighboring* if

$$|x_1 - x_2| \leq 1 \text{ and } |y_1 - y_2| \leq 1.$$

In other words, a vertex of a separator neighbors with any vertex directly to directions N , W , S , E (we call these the *grid directions*), or directly to directions NW , SW , SE , NE (we call these the *intermediate directions*). The *boundary* of the grid consists of the four paths, each having m vertices, with $x = 0$, $x = m - 1$, $y = 0$, and $y = m - 1$, respectively.

Lemma 5.3. *The vertices of any inclusion minimal separator S of G_m , for $m \geq 2$, can be put in a sequence $v_1, v_2, \dots, v_{|S|}$, such that adjacent vertices in the sequence are neighboring and either (i) the first and the last vertex of the sequence are also neighboring, or (ii) if the first and the last vertex of the sequence are not neighboring, then the first and the last vertex of the sequence are the only ones lying on the boundary of the grid.*

Proof. By lemma 5.2 any inclusion minimal separator divides the graph into two connected components and by lemma 5.1, every vertex of the separator is adjacent with a vertex from both connected components. Consider a vertex of the separator which is not on the sides of the grid. This vertex can not have less than two neighboring vertices in the separator, because then all its adjacent vertices in the grid that are not in the separator are in the same connected component.

It is also not possible that a vertex v of the inclusion minimal separator has two neighbors u and w such that both of these neighbors are in the grid directions and the angle uvw in the standard drawing is right. Assume, without loss of generality, that vertex $v = (x, y)$ has neighbors $u = (x, y - 1)$ and $w = (x + 1, y)$ in the separator. Then, by lemma 5.1, since $a = (x, y + 1)$ and $b = (x - 1, y)$ must be in different components, vertex $z = (x - 1, y + 1)$ must also be in the separator (see figure 3). But then, vertex v has three neighbors in the inclusion minimal separator and this is a case that we will prove impossible immediately in the following.

We are now going to prove that in an inclusion minimal separator a vertex v can not have more than two neighbors. We consider different cases on the number of neighbors at intermediate

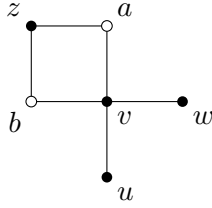


Figure 3: Vertex v with two grid direction neighbors in the separator

directions, ranging from one to three, as shown in figure 4 (we do not have to check the case where three neighbors are all at grid directions, because we have already discussed it). In all cases, the

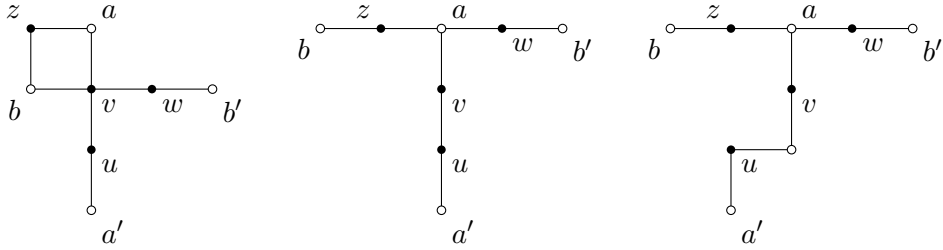


Figure 4: Vertex v has three neighbors of which 1, 2, and 3 are at intermediate directions

impossibility proofs are similar to the proof of lemma 5.2. In particular, with the labeling of vertices shown in figure 4, where a and a' are in the same component A of $G_m - S$ and b and b' are in the same other component B of $G_m - S$, in all three cases, we consider a simple path from a to a' which in the standard drawing of graph G_m is embedded as a simple polygonal line. We also consider the polygonal line $avua'$ which does not intersect the previous polygonal line. The two polygonal lines form a simple closed curve K_a , which (by the Jordan curve theorem) separates the plane into two maximal connected subsets (regions). The polygonal lines vzb and $vw b'$ do not intersect K_a except at v and lie in different regions. Therefore b and b' lie in different regions. Since b and b' are in the same connected component B of $G_m - S$, there is a simple path from b to b' consisting only of vertices in B . This simple path induces a simple curve K_b in the embedding of G_m and, since b and b' are in different regions, this curve intersects the closed curve K_a at some point. Since K_b arises from a path in G_m and K_a is a special polygonal curve with vertices only on grid points, curves K_a and K_b can only intersect at some vertex point, which is a contradiction because curve K_a consists only of vertices in A and S (not B).

The four possible neighboring cases of a vertex in the inclusion minimal separator which does not lie on the sides of the grid, ignoring rotations, are shown in figure 5.



Figure 5: Four possible neighboring cases for an inclusion minimal separator vertex

It is also impossible, in an inclusion minimal separator S , to have a vertex $v \in S$ on the boundary of the grid with only one neighbor $w \in S$, such that w is also on the boundary of the grid, because then v has only neighbors in one connected component of $G_m - S$.

Now, we are going to build a special sequence of vertices of an inclusion minimal separator S . We consider two different cases.

- If there is no vertex with only one neighbor in the separator (case i), then we choose any vertex v as the initial vertex of the sequence. Vertex v has two neighbors in S , say v' and v'' . We choose any of them, say v' , to be the next vertex in the sequence. Then, we extend the sequence by choosing the next element to be the neighbor of the current element not already in the sequence, until we reach the neighbor v'' of v (this always happens because the separator S has a finite number of elements). We claim that the built sequence includes all vertices of separator S . Consider the closed polygonal line K with vertices in the order of the sequence built. This curve K is a simple closed curve (i.e., it does not intersect itself). By the Jordan curve theorem, K divides the plane in a region inside the curve and an unbounded region outside the curve. Both regions contain vertices of the original graph, as can be seen in figure 5 (the neighborhood of at least one vertex of S is as shown in figure 5). Every embedding of a path in G_m connecting two vertices in different regions has to touch closed curve K , and in particular one of its vertices. Thus, the vertices of the built sequence are indeed a separator, and since S is inclusion minimal the built sequence includes all vertices of S .
- If there is a vertex v with only one neighbor in S (case ii), then we choose v as the initial vertex of the sequence (this vertex has to lie on the boundary of the grid). Then, we extend the sequence by choosing the next element to be the neighbor of the current element not already in the sequence, until we reach an element w with only one neighbor in S (this always happens because the separator S has a finite number of elements). We claim that the built sequence includes all vertices of separator S . Consider, the polygonal line K with vertices in the order of the sequence built. With the help of the Jordan curve theorem, one can prove that K divides the square containing the embedding of the grid in two regions. Both regions contain vertices of the original graph, as can be seen in figure 5. Every embedding of a path in G_m connecting two vertices in different regions has to touch closed curve K , and in particular one of its vertices. Thus, the vertices of the built sequence are indeed a separator, and since S is inclusion minimal the built sequence includes all vertices of S .

As a result, inclusion minimal separators are of the following two types, according to their aforementioned built sequence:

- (i) the first and the last vertex of the sequence are also neighboring,
- (ii) if the first and the last vertex of the sequence are not neighboring, then the first and the last vertex of the sequence are the only ones lying on the boundary of the grid.

□

Examples of the two types of inclusion minimal separators are shown in figure 6, together with the polygonal curve induced by their sequence from lemma 5.3.

Remark 5.4. If the curve induced by the sequence of the vertices of an inclusion minimal separator includes both a segment in the $N-S$ direction and a segment in the $E-W$ direction then it must contain a segment going in one of the intermediate directions, $NW-SE$ or $NE-SW$, in order to avoid a vertex v in the separator with two neighbors u and w in the grid directions such that the angle uvw in the standard drawing is right.

Theorem 5.5. For $m \geq 2$, $\chi_{\text{um}}(G_m) \geq \frac{5}{3}m - \log_{\frac{5}{2}} m$.

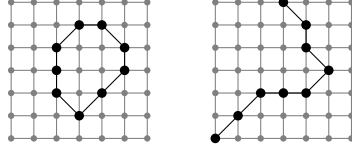


Figure 6: Inclusion minimal separators of types (i) and (ii)

Proof. First, we prove lower bound $\ell(m) = \frac{5}{3}m - \log_{\frac{5}{2}} m$, when $2 \leq m \leq 4$. For $m = 2$, we have $G_2 \supseteq P_4$ and thus, from proposition 1.8, $\chi_{\text{um}}(G_2) \geq \chi_{\text{um}}(P_4) = 3 > \ell(2)$. For $m = 3$ and $m = 4$, in an optimal coloring of G_3 or G_4 it is not possible to have an inclusion minimal separator of type (i), i.e., only an inclusion minimal separator of type (ii) is possible. We will prove that $\chi_{\text{um}}(G_3) \geq 5 > \ell(3)$ and $\chi_{\text{um}}(G_4) \geq 7 > \ell(4)$. Assume, without loss of generality, that the sequence of the inclusion minimal separator starts at column $x = 0$. If the sequence ends at the same column or at any of the rows $y = 0$ or $y = m - 1$, then it is possible to prove that $\chi_{\text{um}}(G_m) \geq 2 + \chi_{\text{um}}(G_{m-1})$, which implies the desired lower bound for $m = 3$ and $m = 4$; in fact we defer this proof for later, for any value of $m \geq 2$, in case (a) below. Otherwise, the sequence of the inclusion minimal separator starts at column $x = 0$ and ends at column $x = m - 1$, without touching the top or bottom rows (which are path P_m subgraphs). In that case, from lemma 1.6, $\chi_{\text{um}}(G_m) \geq m + \chi_{\text{um}}(P_m)$ which implies the desired lower bound for $m = 3$ and $m = 4$, because $\chi_{\text{um}}(P_3) = 2$ and $\chi_{\text{um}}(P_4) = 3$.

Now consider an optimal coloring C of G_m , with $m \geq 5$, and a set of unique colors in C that induce an inclusion minimal separator S .

If the inclusion minimal separator S is of type (i), then consider the diagonals $y + x = i$, where $i \in \{0, \dots, m - 1\}$. Find the minimum i for which diagonal $y + x = i$ contains a vertex of the separator. By checking the possible configurations of vertices in the separator, we can prove that there is a vertex $(x, y) \in S$ with maximum y in the diagonal $y + x = i$ such that $(x + 1, y - 1)$ is also in S and one of $(x, y + 1)$, $(x + 1, y + 1)$ is also in S , and there is a different vertex $(x', y') \in S$ with minimum y' in the diagonal $y + x = i$ such that $(x' - 1, y' + 1)$ is also in S and one of $(x' + 1, y')$, $(x' + 1, y' + 1)$ is also in S . Then one can contract edges of G_m as shown in figure 7 (for each gray area of the figure, all contained vertices in the area are contracted to a single vertex), in order to get a graph G' which contains G_{m-1} as a subgraph. Observe that in the square gray area of figure 7 two vertices of the separator are contained, and the same is true for one of the triangular gray areas (the lowest and rightmost triangular area). As a result, we have a coloring C' for G' using two fewer colors than an optimal coloring of G_m and thus $\chi_{\text{um}}(G_m) \geq \chi_{\text{um}}(G') + 2 \geq \chi_{\text{um}}(G_{m-1}) + 2 \geq \frac{5}{3}(m - 1) - \log_{\frac{5}{2}}(m - 1) + 2 > 5m/3 - \log_{\frac{5}{2}} m$.

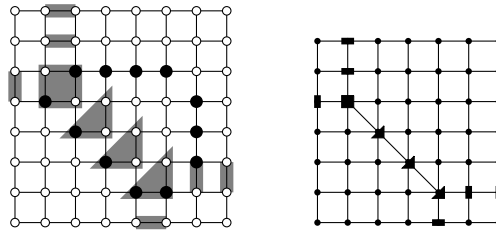


Figure 7: Graph G_m and its minor G' containing G_{m-1}

If the inclusion minimal separator is of type (ii), then we have two cases.

In the first case (a), without loss of generality, the sequence of the inclusion minimal separator S starts at vertex u of column $x = 0$ and ends at vertex v which is in column $x = 0$ or in row $y = 0$.

Then, $G - \{u, v\} \supseteq G_{m-1}$, because all columns and rows of G_m , except column $x = 0$ and row $y = 0$, are intact in $G - \{u, v\}$, and thus from lemma 1.6 $\chi_{\text{um}}(G_m) \geq \chi_{\text{um}}(G_{m-1}) + 2 \geq \ell(m-1) + 2 > \ell(m)$. Observe that this proof is good for any value of $m \geq 2$.

In the second case (b), without loss of generality, the sequence of the inclusion minimal separator S starts at column $x = 0$ and ends at column $x = m - 1$, touching neither row $y = 0$ nor row $y = m - 1$. Observe that in that case, $|S| \geq m$. We consider two subcases, either $|S| > m$, or $|S| = m$.

If $|S| > m$, then consider the sequence of the vertices of the inclusion minimal separator $v_1, v_2, \dots, v_{|S|}$, with v_1 in column $x = 0$ and $v_{|S|}$ in column $x = m - 1$. Since $|S| > m$, there must exist some $v_i = (x, y)$, such that, without loss of generality, $v_{i+1} = (x, y - 1)$ or $v_{i+1} = (x - 1, y - 1)$. Consider the point v_i with the above property such that i is minimum.

If the polygonal curve v_1, v_2, \dots, v_i is not a straight line segment with slope $-\pi/4$, then there is some $i' < i$ such that $v_{i'} = (x', y')$ and either $v_{i'-1} = (x' - 1, y')$ or $v_{i'-1} = (x' - 1, y' - 1)$. Consider the point $v_{i'}$ with the above property such that i' is maximum. Then, contracting as shown in figure 8a (note the similarity with the contraction when the separator is of type i), will result in a coloring C' of the resulting graph $G' \supseteq G_{m-1}$ using two fewer colors than an optimal coloring of G_m and thus $\chi_{\text{um}}(G_m) \geq \chi_{\text{um}}(G') + 2 \geq \chi_{\text{um}}(G_{m-1}) + 2 \geq \frac{5}{3}(m-1) - \log_{\frac{5}{2}}(m-1) + 2 > 5m/3 - \log_{\frac{5}{2}} m$.

If the polygonal curve v_1, v_2, \dots, v_i is a straight line segment with slope $-\pi/4$, then since $v_{|S|}$ is on column $m - 1$, there must be some $v_j = (x, y)$, with $i < j$, such that either $v_{j+1} = (x + 1, y)$ or $v_{j+1} = (x + 1, y + 1)$. Consider the point v_j with the above property such that j is minimum. Moreover, there is some $v_{j'} = (x', y')$, with $i < j' < j$, such that either $v_{j'-1} = (x', y' + 1)$ or $v_{j'-1} = (x' + 1, y' + 1)$. Consider the point $v_{j'}$ with the above property such that j' is maximum. Then, contracting as shown in figure 8b (note the similarity with the contraction when the separator is of type i), will result in a coloring C' of the resulting graph $G' \supseteq G_{m-1}$ using two fewer colors than an optimal coloring of G_m and thus $\chi_{\text{um}}(G_m) \geq \chi_{\text{um}}(G') + 2 \geq \chi_{\text{um}}(G_{m-1}) + 2 \geq \frac{5}{3}(m-1) - \log_{\frac{5}{2}}(m-1) + 2 > 5m/3 - \log_{\frac{5}{2}} m$.

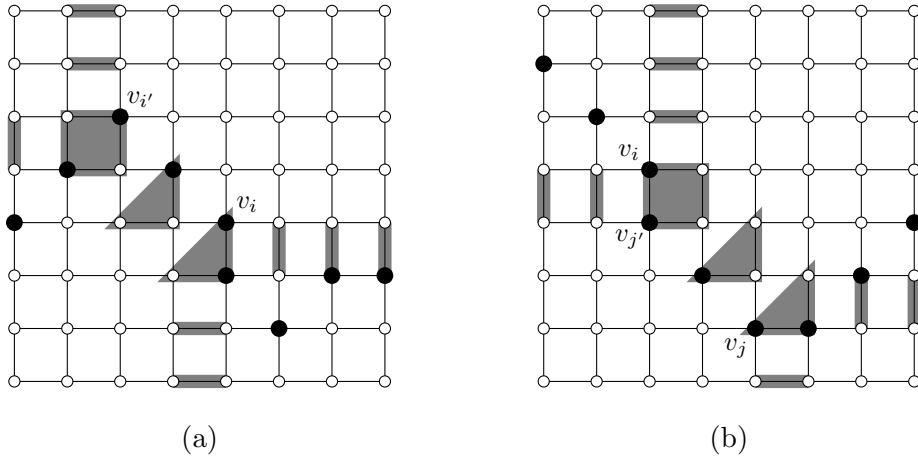


Figure 8: The subcase $|S| > m$

If $|S| = m$, then the sequence of vertices v_1, v_2, \dots, v_m is such that $v_i = (i, y_i)$, for every i .

If $m = 5k$, where k is a positive integer, we will prove that $G_m - S \supseteq G_{2k}$. Consider the set of vertices

$$A = \{(x, y) \mid k \leq x \leq 4k - 1, 0 \leq y \leq 2k - 1\}.$$

Set A induces a $3k \times 2k$ grid graph, $G_{3k, 2k}$, in G_m . If $A \cap S = \emptyset$, then $G_m - S \supseteq G_{3k, 2k} \supseteq G_{2k}$;

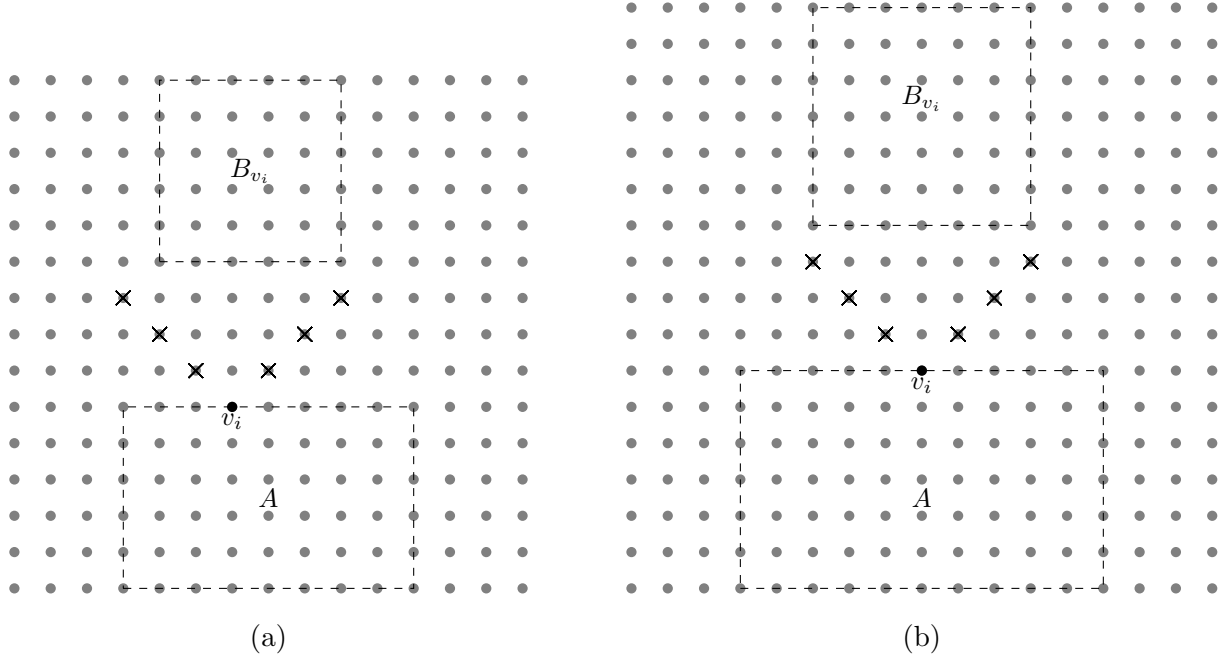


Figure 9: The subcase $|S| = m$, grids with $m = 5k$ and $m = 5k + 2$

otherwise some $v_i \in S$ belongs to A , i.e., $v_i = (i, y_i)$ with $k \leq i \leq 4k - 1$ and $0 \leq y_i \leq 2k - 1$. Then, consider the set of vertices

$$B_{v_i} = \{(x, y) \mid i - k + 1 \leq x \leq i + k, 3k \leq y \leq m - 1\},$$

which induces a G_{2k} subgraph in G_m . Adjacent vertices of the separator are neighboring, therefore, for a vertex $(x', y') \in S$, we have $|y' - y_i| \leq |x' - i|$. But for vertices of S in the strip $i - k + 1 \leq x \leq i + k$, we have $|x' - i| \leq k$, and thus $|y' - y_i| \leq k$, which implies $y' - y_i \leq k$ and $y' \leq 3k - 1$ (because $y_i \leq 2k - 1$). Therefore, $B_{v_i} \cap S = \emptyset$ and thus $G_m - S \supseteq G_{2k}$ (see figure 9b).

If $m = 5k + 2$, where k is a positive integer, we will prove that $G_m - S \supseteq G_{2k+1}$. Consider the set of vertices

$$A = \{(x, y) \mid k \leq x \leq 4k + 1, 0 \leq y \leq 2k\}.$$

Set A induces a $G_{3k+2, 2k+1}$ in G_m . If $A \cap S = \emptyset$, then $G_m - S \supseteq G_{3k+2, 2k+1} \supseteq G_{2k+1}$; otherwise some $v_i \in S$ belongs to A , i.e., $v_i = (i, y_i)$ with $k \leq i \leq 4k + 1$ and $0 \leq y_i \leq 2k$. Then, consider the set of vertices

$$B_{v_i} = \{(x, y) \mid i - k \leq x \leq i + k, 3k + 1 \leq y \leq m - 1\},$$

which induces a G_{2k+1} subgraph in G_m . Adjacent vertices of the separator are neighboring, therefore, for a vertex $(x', y') \in S$, we have $|y' - y_i| \leq |x' - i|$. But for vertices of S in the strip $i - k \leq x \leq i + k$, we have $|x' - i| \leq k$, and thus $|y' - y_i| \leq k$, which implies $y' - y_i \leq k$ and $y' \leq 3k$ (because $y_i \leq 2k$). Therefore, $B_{v_i} \cap S = \emptyset$ and thus $G_m - S \supseteq G_{2k+1}$ (see figure 9b).

In the subcase $|S| = m$, with $m \geq 5$, we have proven that $G_m - S$ contains a square grid

subgraph $G_{g(m)}$, where

$$g(m) = \begin{cases} 2k, & \text{if } m = 5k, \\ 2k, & \text{if } m = 5k + 1, \\ 2k + 1, & \text{if } m = 5k + 2, \\ 2k + 1, & \text{if } m = 5k + 3, \\ 2k + 1, & \text{if } m = 5k + 4, \end{cases} = \begin{cases} \frac{2}{5}m, & \text{if } m \bmod 5 = 0, \\ \frac{2}{5}m - \frac{2}{5}, & \text{if } m \bmod 5 = 1, \\ \frac{2}{5}m + \frac{1}{5}, & \text{if } m \bmod 5 = 2, \\ \frac{2}{5}m - \frac{1}{5}, & \text{if } m \bmod 5 = 3, \\ \frac{2}{5}m - \frac{3}{5}, & \text{if } m \bmod 5 = 4. \end{cases}$$

As a result, $\chi_{\text{um}}(G_m) \geq |S| + \chi_{\text{um}}(G_{g(m)}) \geq m + \ell(g(m))$, by induction. However, since $\ell(x)$ is monotone increasing for real $x \geq 5$, we have $\ell(g(m)) \geq \ell(\frac{2}{5}m - \frac{3}{5})$ and therefore $\chi_{\text{um}}(G_m) \geq m + \ell(\frac{2}{5}m - \frac{3}{5}) = \frac{5}{3}m - \log_{\frac{5}{2}}(m - \frac{3}{2}) \geq \frac{5}{3}m - \log_{\frac{5}{2}}m$. \square

An immediate corollary from theorem 4.8 is the following.

Corollary 5.6. *For $m \geq 2$, $\chi_{\text{cf}}(G_m) \geq \frac{5}{6}m - \frac{1}{2} \log_{\frac{5}{2}}m$.*

6 Discussion and open problems

As we mentioned in the introduction, conflict-free and unique-maximum colorings can be defined for hypergraphs. In the literature of conflict-free coloring, hypergraphs that are induced by geometric shapes have been in the focus. It would be interesting to show possible relations of the respective chromatic numbers in this setting.

An interesting open problem is finding the exact value of the unique-maximum chromatic number for the square grid G_m . In this paper, we improved the lower bound asymptotically to $5m/3$, but the best known upper bound from [1, 2] is about $2.514m$. We believe that a study of lower bounds for rectangular (i.e., non-square) grids can help improve the lower bound for the square grid.

Another area for improvement is the relation between the two chromatic numbers for general graphs. We have only found graphs which have unique-maximum chromatic number about twice the conflict-free chromatic number, but the only bound we have proved on $\chi_{\text{um}}(G)$ is exponential in $\chi_{\text{cf}}(G)$.

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