Unique-maximum and conflict-free colorings for hypergraphs and tree graphs

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Abstract

We investigate the relationship between two kinds of vertex colorings of hypergraphs: unique-maximum colorings and conflict-free colorings. In a unique-maximum coloring, the colors are ordered, and in every hyperedge of the hypergraph the maximum color appears only once. In a conflict-free coloring, in every hyperedge of the hypergraph there is a color that appears only once. We concentrate in hypergraphs that are induced by paths in tree graphs.

Keywords: unique-maximum coloring, conflict-free coloring, hypergraph, tree graph

1 Introduction

In this paper we study some special vertex colorings of a hypergraph $H = (V, E)$, i.e., functions $C\colon V \to \mathbb{N}^+$. Since a hypergraph is a generalization of a graph, it is natural to consider how to generalize traditional vertex coloring of a graph (in which two vertices neighboring with an edge in the graph have to be assigned different colors by the function C) to a vertex coloring of a hypergraph. Vertex coloring in hypergraphs can be defined in many ways, so that restricting the definition to simple graphs coincides with traditional graph coloring. At one extreme, it is only required that the vertices of each hyperedge are not all colored with the same color (except for singleton hyperedges). This is called a *non-monochromatic* coloring of a hypergraph. The minimum number of colors necessary to color in such a way a hypergraph H is the (non-monochromatic) chromatic number of H, denoted by $\chi(H)$. At the other extreme, we can require that the vertices of each hyperedge are all colored with different colors. This is called a *colorful* or *rainbow* coloring of H and we have the corresponding rainbow chromatic number of H, denoted by $\chi_{\mathrm{rb}}(H)$. In this paper we study three types of vertex colorings of hypergraphs that are between the above two extremes. The first two have been studied before [7]; the third is new, we use it in our proofs, and it could also be of independent interest.

Definition 1.1. A *unique-maximum coloring* of $H = (V, E)$ with k colors is a function $C\colon V \to \{1, \ldots, k\}$ such that for each $e \in E$ the *maximum color occurs exactly once* on the vertices of e. The minimum k for which a hypergraph H has a unique-maximum coloring with k colors is called the *unique-maximum chromatic number* of H and is denoted by $\chi_{\mathrm{um}}(H)$.

Definition 1.2. A *conflict-free coloring* of $H = (V, E)$ with k colors is a function $C\colon V \to \{1, \ldots, k\}$ such that for each $e \in E$ there is *a color that occurs exactly once* on the vertices of e. The minimum k for which a hypergraph H has a conflict-free coloring with k colors is called the *conflict-free chromatic number* of H and is denoted by $\chi_{\mathrm{cf}}(H)$.

Definition 1.3. An *odd coloring* of $H = (V, E)$ with k colors is a function $C: V \to \{1, \ldots, k\}$ such that for each $e \in E$ there is *a color that occurs an odd number of times* on the vertices of e. The minimum k for which a hypergraph H has an odd coloring with k colors is called the *odd chromatic number* of H and is denoted by $\chi_{\text{odd}}(H)$.

Every rainbow coloring is unique-maximum, every unique-maximum coloring is conflict-free, and every conflict-free coloring is odd and non-monochromatic. Therefore, for every hypergraph H, $\max(\chi(H), \chi_{\text{odd}}(H)) \leq \chi_{\text{cf}}(H) \leq \chi_{\text{um}}(H) \leq \chi_{\text{rb}}(H)$. Note that an odd coloring can be monochromatic.

The study of conflict-free coloring hypergraphs started in [7], with an emphasis in hypergraphs induced by geometric shapes. The main application of conflict-free coloring is that it represents a frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile agents*. Base stations have fixed positions and provide the backbone of the network; they are represented by vertices in V. Mobile agents are the clients of the network and they are served by base stations. This is done as follows: Every base station has a fixed frequency; this is represented by the coloring C, i.e., colors represent frequencies. If an agent wants to establish a link with a base station it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not used by some other base station in the range. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, where possible, is preferable. Conflict-free coloring problems have been the subject of many recent papers due to their practical and theoretical interest (see e.g. [16, 8, 5, 6, 1]). Most approaches in the conflict-free coloring literature use unique-maximum colorings (a notable exception is the 'triples' algorithm in [1]), because unique-maximum colorings are easier to argue about in proofs, due to their additional structure. Another advantage of unique-maximum colorings is the simplicity of computing the unique color in any range (it is always the maximum color), given a unique-maximum coloring, which can be helpful if very simple mobile devices are used by the agents.

Other hypergraphs that have been studied with respect to these colorings, are ones which are induced by a graph and its neighborhoods or its paths. In particular, given a graph G, consider the hypergraph with the same vertex set as G and a hyperedge for every distinct vertex neighborhood of G; such conflict-free colorings have been studied in [3, 15]. Alternatively, given a graph G, consider the hypergraph H with the same vertex set as G and a hyperedge for every distinct vertex set that can be spanned by a path of G. A unique-maximum (resp. conflict-free, odd) coloring of H is called a unique-maximum (resp. conflict-free, odd) coloring of G with respect to paths; we also define the corresponding graph chromatic numbers, $\chi^{\text{p}}_{\text{um}}(G) = \chi_{\text{um}}(H)$, $\chi^{\text{p}}_{\text{cf}}(G) = \chi_{\text{cf}}(H)$ and $\chi^{\text{p}}_{\text{odd}}(G) = \chi_{\text{odd}}(H)$. Sometimes to improve readability of the text, we simply talk about the UM (resp. CF, ODD) of a graph. Unique maximum colorings with respect to paths of graphs are known alternatively in the literature as *ordered colorings* or *vertex rankings*, and are closely related to *tree-depth* [14]. The problem of computing such unique-maximum colorings is a well-known and widely studied problem (see e.g. [10]) with many applications including VLSI design [11] and parallel Cholesky factorization of matrices [12]. The problem is also interesting for the Operations Research community, because it has applications in planning efficient assembly of products in manufacturing systems [9]. In general, it seems that the vertex ranking problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.) For general graphs, finding the exact unique-maximum chromatic number with respect to paths of a graph is NP-complete [17, 13] and there is a polynomial time

$O(\log^2 n)$ approximation algorithm [2], where n is the number of vertices. The paper [4] studied the relationship between the two graph chromatic numbers, $\chi^{\mathrm{p}}_{\mathrm{um}}(G)$ and $\chi^{\mathrm{p}}_{\mathrm{cf}}(G)$. The authors proved that even checking whether a given coloring of a graph is conflict-free is coNP-complete (whereas the same problem is in P for unique-maximum colorings). Moreover, they showed that for every graph G, $\chi^{\mathrm{p}}_{\mathrm{um}}(G) \leq 2^{\chi^{\mathrm{p}}_{\mathrm{cf}}(G)}-1$ and provided a sequence of graphs for which the ratio $\chi^{\mathrm{p}}_{\mathrm{um}}(G)/\chi^{\mathrm{p}}_{\mathrm{cf}}(G)$ tends to 2.

In this work, we study the relationship between unique-maximum and conflict-free colorings. First we give an exact answer to the question "How larger than $\chi_{\mathrm{cf}}(H)$ can $\chi_{\mathrm{um}}(H)$ be?" for a general hypergraph H showing that this gap can be big. Then we turn to hypergraphs induced by paths in graphs and prove a better bound for $\chi^{\mathrm{p}}_{\mathrm{cf}}(T)$ and $\chi^{\mathrm{p}}_{\mathrm{cf}}(T)$, where T is a tree graph. The reason we attempt to answer the question for trees is that the upper and lower bound for general graphs from [4] are quite far apart. Indeed, for trees we manage to prove upper and lower bounds on the difference of $\chi^{\mathrm{p}}_{\mathrm{cf}}(T)$ and $\chi^{\mathrm{p}}_{\mathrm{cf}}(T)$ that are closer than the ones for general graphs.

Paper organization. In section 2, we show that if for a hypergraph H, $\chi_{\mathrm{cf}}(H) = k > 1$, then $\chi_{\mathrm{um}}(H)$ is bounded from above, roughly, by $\frac{k-1}{k}|V|$, and this is tight; the result remains true even if we restrict ourselves to uniform hypergraphs. In section 3, we show that for every tree graph T, $\chi^{\mathrm{p}}_{\mathrm{um}}(T) \leq (\chi^{\mathrm{p}}_{\mathrm{cf}}(T))^3$ and provide a sequence of trees for which the ratio $\chi^{\mathrm{p}}_{\mathrm{um}}(T)/\chi^{\mathrm{p}}_{\mathrm{cf}}(T)$ tends to a constant c with $1 < c < 2$. Conclusions and open problems are presented in section 4.

1.1 Preliminaries

Proposition 1.4. *Each of the graph chromatic numbers $\chi^{\mathrm{p}}_{\mathrm{um}}$, $\chi^{\mathrm{p}}_{\mathrm{cf}}$, and $\chi^{\mathrm{p}}_{\mathrm{odd}}$, is monotone with respect to subgraphs, i.e., if $H \subseteq G$, then $\chi^{\mathrm{p}}_{\diamond}(H) \leq \chi^{\mathrm{p}}_{\diamond}(G)$, where $\diamond \in \{\mathrm{um}, \mathrm{cf}, \mathrm{odd}\}$.*

Proof. A subgraph H of a graph G contains a subset of the paths of G. □

Definition 1.5 (Parity vector). Given a coloring $C\colon V \to \{1, \ldots, k\}$ and a set $e \subseteq V$, the *parity vector* of e is the vector of length k in which the i^{th} coordinate equals the parity (0 or 1) of the number of elements in e colored with i.

Remark 1.6. A coloring of a hypergraph is odd if and only if the parity vector of every hyperedge is not the all-zero vector.

2 The two chromatic numbers for general hypergraphs

In general it is not possible to bound χ_{cf} with a function of χ_{odd} because if we take our hyperedges to be all triples of $\{1, \ldots, n\}$, for the resulting hypergraph H we have $\chi_{\mathrm{odd}}(H) = 1$ and $\chi_{\mathrm{cf}}(H) = \lceil \frac{n}{2} \rceil$. Although $\chi_{\mathrm{cf}}(H) = 1$ implies $\chi_{\mathrm{um}}(H) = 1$, we can have a big gap as is shown by the following theorem.

Theorem 2.1. *For an arbitrary hypergraph H on n vertices, $\chi_{\mathrm{um}}(H) \leq n - \lceil n/\chi_{\mathrm{cf}}(H) \rceil + 1$. Moreover, this is the best possible bound, i.e., for arbitrary n there exists a hypergraph for which equality holds.*

Proof. A simple algorithm achieving the upper bound is the following. Given a H with $\chi_{\mathrm{cf}}(H) = k$, take a conflict-free coloring of H with k colors, color the biggest color class with color 1, all the other vertices with all different colors (bigger than 1). It is not difficult to see that this is a unique-max coloring, and it uses at most $n - \lceil n/k \rceil + 1$ colors.

For a given n and k equality holds for the hypergraph H whose n vertices are partitioned into k almost equal parts, all of size $\lceil n/k \rceil$ or $\lceil n/k \rceil - 1$ and its edges are all sets of size 2 and 3 covering vertices from exactly 2 parts.

For this graph we have $\chi_{\text{cf}}(H) = k$ because in any conflict-free coloring of H there are no two vertices in different parts having the same color and $\chi_{\text{um}}(H) \leq n - \lceil n/k \rceil + 1$ because in any unique-max coloring of H all vertices must have different colors except the vertices of one part. □

For hypergraphs without small hyperedges, we can make the inequality tighter.

Theorem 2.2. *If $l \geq 3$ then for an arbitrary l-uniform hypergraph H with $\chi_{\text{cf}}(H) = k$ having $n \geq 2kl$ vertices we have $\chi_{\text{um}}(H) \leq n - \lceil n/k \rceil - l + 4$. Moreover, this is the best possible bound, i.e., for arbitrary $n \geq 2kl$ there exists a hypergraph for which equality holds.*

The proof is similar to the previous, although longer, and can be found in the Appendix.

3 The two chromatic numbers for trees

In this section to ease readability we use UM for $\chi^{\text{p}}_{\text{um}}$, CF for $\chi^{\text{p}}_{\text{cf}}$ and ODD for $\chi^{\text{p}}_{\text{odd}}$. We denote by P_n the path graph with n vertices. As a warm-up we prove the following simple claim. (Note that log always denotes the logarithm of base 2.)

Claim 3.1. *For $n \geq 1$, $UM(P_n) = CF(P_n) = ODD(P_n) = \lceil \log(n+1) \rceil$.*

Proof. It is easy to see that $UM(P_n) \leq \lceil \log(n+1) \rceil$, just give the biggest color to (one of) the central vertex(es). Since we know that $UM(P_n) \geq CF(P_n) \geq ODD(P_n)$, it is enough to prove that $2^{ODD(P_n)} > n$. Take the n paths starting from one endpoint. If there were two with the same parity vector, their symmetric difference (which is also a path) would contain an even number of each color. Thus we have at least n different parity vectors, none of which is the all-zero vector. But the number of non-zero parity vectors is at most $2^{ODD(P_n)} - 1$. □

3.1 Upper bound for binary trees

We denote by B_d the (rooted) complete binary tree with d levels (and $2^d - 1$ vertices). We prove a quadratic upper bound on the gap of complete binary trees. In fact, we will prove a stronger statement, a bound on the gap of UM and ODD. It is easy to see that $UM(B_d) = d$; for an optimal unique-maximum coloring, color the leaves of B_d with color 1, their parents with color 2, and so on, until you color the root with color d; for a matching lower bound, use induction on d.

Definition 3.2. A graph H is a *subdivision* of G if H is obtained by substituting every edge of G by a path. The original vertices of G in H are called *branch vertices.*

Given a rooted tree T with root vertex r, consider the distance $d_r(v)$ of every vertex v from the root. We say that a subdivision B^* of B_d contained in T is *compatible with* T, if for every branch vertex v and for every branch vertex v' which is a descendant of v in the natural ordering of the B_d tree, we have $d_r(v) < d_r(v')$.

We first need the following lemma.

Lemma 3.3. *If we color with k colors (without any restrictions) the vertices of a rooted tree T containing a compatible to T subdivision B^* of B_d, then there exists a vector $a = (a_1, a_2, ..a_k)$ such that $\sum_{i=1}^{k} a_i = d$ and for every $i \in \{1, \ldots, k\}$, B^* contains a compatible to T subdivision of B_{a_i} whose branch vertices are all colored with i.*

Proof. By induction on d. Consider the branch vertex v of B^* that corresponds to the root of B_d. If deleting v gives two different vectors in the two subtrees, then we can take their coordinate-wise maximum. If the vectors are the same, we can increase a_i for the color i of v. □

Theorem 3.4. *For $d \geq 1$ and for every subdivision B^* of B_d, $ODD(B^*) \geq \sqrt{d}$.*

Proof. Fix an optimal odd coloring with k colors. Fix an i for which in lemma 3.3 we have $a_i \geq \frac{d}{k}$.

Consider the 2^{a_i-1} paths that originate in a leaf of the B_{a_i} subdivision and end in its root branch vertex. We claim that the parity vectors of the 2^{a_i-1} paths must be all different. Indeed, if there were two paths with the same parity vector, then the symmetric difference of the paths plus their lowest common vertex would form a path where the parity of each color is even, except maybe for color i, but since this new path starts and ends with color i, deleting any of its ends yields a path whose parity vector is the all-zero vector, a contradiction.

There are at most $2^k - 1$ parity vectors, thus $2^k - 1 \geq 2^{a_i-1} \geq 2^{\lceil d/k \rceil - 1}$. From this we get $k > \lceil \frac{d}{k} \rceil - 1$ which is equivalent to $k \geq \lceil \frac{d}{k} \rceil$ using the integrality, thus $k \geq \sqrt{d}$. □

3.2 Upper bound for arbitrary trees

We will try to find either a long path or a deep binary tree in every tree with high UM. For this, we need the notion of *UM*-critical trees and their characterization from [10].

Definition 3.5. A graph is *UM-critical*, if the UM of any of its subgraphs is smaller than its UM.

We also say that a graph is *k-UM-critical*, if it is UM-critical and its UM equals k.

Example 3.6. K_k and the path with 2^{k-1} vertices are both k-UM-critical. For $k \leq 3$ there is a unique k-UM-critical tree, the path with 2^{k-1} vertices. The graph with 8 vertices that is obtained by connecting two middle vertices of two paths with 4 vertices is 4-UM-critical and its CF is 3. (This is the smallest tree where the CF and UM chromatic numbers differ.)

Theorem 3.7 (Theorem 2.1 in [10])**.** *For $k > 1$, a tree is k-UM-critical if and only if it has an edge that connects two $(k-1)$-UM-critical trees.*

The proof of the theorem can be found in [10] or the interested reader can devise it herself.

Remark 3.8. A k-UM-critical tree has exactly 2^{k-1} vertices and the connecting edge must always be the central edge of the tree. This implies that there is a unique way to partition the vertices of the k-UM-critical tree to two sets of vertices, each inducing a $(k-1)$-UM-critical tree, and so on.

Now we can define the *structure trees* of UM-critical trees.

Definition 3.9. For $1 \leq l \leq k$, the *l-deep structure tree* of a k-UM-critical tree is the tree graph with a vertex for every one of the 2^l $(k-l)$-UM-critical subtrees that we obtain by repeatedly applying theorem 3.7, and an edge between two vertices if the corresponding $(k-l)$-UM-critical subtrees have an edge between them in the k-UM-critical tree.

Example 3.10. The 1-deep structure tree of any UM-critical tree is an edge. The 2-deep structure tree of any UM-critical tree is a path with 4 vertices. The k-deep structure tree of a k-UM-critical tree is itself.

Remark 3.11. It is not difficult to prove that the l-deep structure tree of a UM-critical tree is an l-UM-critical tree.

We start with a few simple observations.

Proposition 3.12. *If an $(l+1)$-UM-critical tree has no vertex of degree at least 3, then it is the path with 2^l vertices.*

Proof. Delete the central edge and use induction. □

Proposition 3.13. *If an $(l+2)$-UM-critical tree has only one vertex of degree at least 3, then it contains a path with 2^l vertices that ends in this vertex.*

Proof. After deleting its central edge, one of the resulting $(l+1)$-UM-critical trees must be a path that was connected to the rest of the graph with one of its ends, thus we can extend it until the high degree vertex. □

Proposition 3.14. *If a tree contains two non-adjacent vertices with degree at least 3, then it contains a subdivision of B_3.*

Proof. The non-adjacent degree 3 vertices will be the second level of the binary tree, and any vertex on the path connecting them the root. □

Claim 3.15. *An $(l+2)$-UM-critical tree contains a path with 2^l vertices or a subdivision of B_3.*

Proof. Because of the previous propositions, we can suppose that our tree has exactly two vertices with degree at least 3 and these are adjacent. If the central edge is not the one between these vertices, then the graph must contain an $(l+1)$-UM-critical subgraph without any vertex with degree at least 3, thus it is the path with 2^l vertices because of proposition 3.12. If it connects the two high degree vertices, then, using proposition 3.13, we have two paths with 2^{l-1} vertices in the $(l+1)$-UM-critical subgraphs obtained by deleting the central edge ending in these vertices, thus with the central edge they form a path with 2^l vertices. □

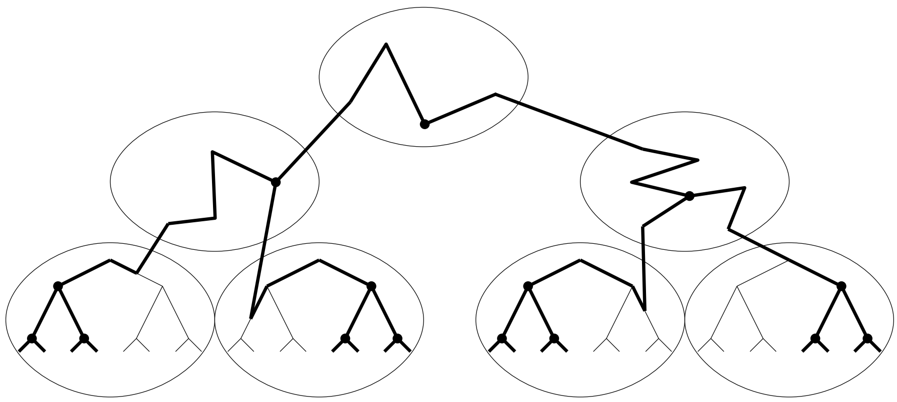

Figure 1: Constructing big binary trees using induction for structure trees.

We are now ready to prove our main lemma, before the proof of the upper bound.

Lemma 3.16. *Every k-UM-critical tree contains a path with 2^l vertices or a subdivision of $B_{\lceil \frac{k+l+3}{l+2} \rceil}$, if $k \geq 3$.*

Proof. The proof is by induction on k. For $3 \leq k \leq l+1$, the statement is true since $B_2 = P_3$. For $l+2 \leq k \leq 2l+3$, the statement is equivalent to our claim 3.15. For $k > 2l+3$, take the $(l+2)$-deep structure tree S of the tree. If S does not contain a path with 2^l vertices, then, using claim 3.15, S contains a subdivision of B_3. Every one of the four leaf branch vertices of the above B_3 subdivision corresponds to a $(k-l-2)$-UM-critical subtree of the original tree. By induction, each one of the above four subtrees must contain a path with 2^l vertices or a subdivision of the complete binary tree with $\left\lceil \frac{k-l-2+l+3}{l+2} \right\rceil = \left\lceil \frac{k+l+3}{l+2} \right\rceil - 1$ levels. If any of them contains the path, we are done. If each one of them contains a $B_{\lceil \frac{k+l+3}{l+2} \rceil -1}$ subdivision, then for every one of the four leaves, we can connect at least one of the two disjoint $B_{\lceil \frac{k+l+3}{l+2} \rceil -2}$ subdivisions of the $B_{\lceil \frac{k+l+3}{l+2} \rceil -1}$ subdivision in the leaf (as in the figure, where each of the four relevant $B_{\frac{k+l+3}{l+2}-2}$ subdivisions and the paths connecting them are shown with heavier lines) to obtain a subdivision of a complete binary tree with $\left\lceil \frac{k+l+3}{l+2} \right\rceil - 2 + 2$ levels, thus we are done. □

Theorem 3.17. *For every tree T, $ODD(T) \geq (UM(T))^{\frac{1}{3}} - O(1)$.*

Proof. If $UM(T) = k$, then T contains a k-UM-critical tree, which from lemma 3.16 contains a P_{2^l} or a subdivision B^* of $B_{\lceil \frac{k+l+3}{l+2} \rceil}$. Using monotonicity of ODD with respect to subgraphs (proposition 1.4), together with $ODD(P_{2^l}) = l+1$ (claim 3.1) and $ODD(B^*) \geq \sqrt{\frac{k+l+3}{l+2}}$ (from theorem 3.4), we get $ODD(T) \geq \max\left(l+1, \sqrt{\frac{k+l+3}{l+2}}\right)$. Choosing l to be the closest integer to the solution of $l+1 = \sqrt{\frac{k+l+3}{l+2}}$, we get $l = k^{\frac{1}{3}} + \Theta(1)$. Therefore, $ODD(T) \geq (UM(T))^{\frac{1}{3}} - O(1)$. □

3.3 Lower bound for binary trees

We have seen that $UM(B_d) = d$. We intend to show conflict-free colorings of some complete binary trees that use substantially less colors. We start with a simple example demonstrating our method.

Claim 3.18. *$CF(B_7) \leq 6$.*

Proof. Color the root with 1, the second level with 2. Deleting the colored vertices leaves four B_5 subtrees. In each of these subtrees, every level will be monochromatic. From top to bottom, in the first use the colors $3,4,5,1,2$, in the second $4,5,6,1,2$, in the third $5,6,3,1,2$ and in the forth $6,3,4,1,2$. It is not difficult to verify that this is indeed a conflict-free coloring (but it will also follow from later results). Observe that in the top 2 levels 2 colors are used, in the next 3 levels 4 colors, and in the last 2 levels the same 2 colors are used as the ones in the top level. □

Corollary 3.19. *$CF(B_{2(r+1)+3r}) \leq 4r+2$.*

Proof. In the previous construction, every leaf had color 2 and their parents had color 1. Every such three vertex part can be the top of a new tree, similar to the original, and replacing $3,4,5,6$ with four new colors. This gives a tree with 12 levels and 10 colors. It is not difficult to verify that this is indeed a conflict-free coloring (but it will also follow from later results). Repeatedly applying this procedure, so that we have colors 1, 2 appearing in $2(r+1)$ levels and r disjoint sets of 4 colors each, we get a coloring of $B_{2(r+1)+3r}$ using $4r+2$ colors. □

To examine more closely why these colorings are conflict-free, we need to define some notions.

Definition 3.20. A family $\mathcal{F}$ of ordered sets is said to be *prefix set-free*, if any prefix of any ordered set is different from any other ordered set as a set (without the ordering). If the ground set has n elements, every sequence has length at least k and the cardinality of $\mathcal{F}$ is at least 2^d, then we say that $\mathcal{F}$ is a $[k,d,n]$ PSF family.

Example 3.21. $\{1,3\}$,$\{1,2,3\}$ is a $[2,1,3]$ PSF family and $\{1\}$,$\{2,1\}$,$\{2,3\}$,$\{3,1\}\{3,1,2\}$ is a $[1,2,3]$ PSF family but $\{2,1\}$,$\{1,2,3\}$ is not a PSF family.

Claim 3.22. *For any $[k,d,n]$ PSF family $d \leq \log \sum_{i=k}^{n} \binom{n}{i}$.*

Proof. Any two ordered sets must differ as sets. □

Claim 3.23. *There is a $[k,d,n]$ PSF family with $d = \lfloor \log \binom{n}{k} \rfloor$.*

Proof. Take all k element subsets of $\{1,\dots,n\}$ and order each arbitrarily. □

Since these bounds do not differ much if $k > (\frac{1}{2}+\epsilon)n$, we do not attempt to get sharper bounds.

Theorem 3.24. *If there is a $[k,d,n]$ PSF family where the size of every set is at most $k+d$, then $CF(B_{d(r+1)+kr}) \leq nr + d$.*

Proof. First, we show that $CF(B_{k+2d}) \leq n + d$. Color the top d levels with d colors. Remove the colored vertices and consider the 2^d B_{k+d} subtrees left. To each associate an ordered set from the $[k,d,n]$ PSF family and color the whole i^{th} level with one color, the i^{th} element of the associated ordered set. Deleting also these colored vertices, we are left with subtrees with at most d levels, which we can color with (at most) the same d colors we used for the top levels.

By repeating the above procedure r times for $B_{d(r+1)+kr}$, as in corollary 3.19, we obtain $CF(B_{d(r+1)+kr}) \leq nr + d$. □

Corollary 3.25. *For the sequence of complete binary trees, $\{B_i\}_{i=1}^{\infty}$, the limit of the ratio of the UM to the CF chromatic number is at least $\log 3 \approx 1.58$.*

The existence of the limit is left as an easy exercise to the interested reader.

Proof of corollary 3.25. Since $CF(B_{d(r+1)+kr}) \leq nr + d$, the ratio of UM to CF for $B_{d(r+1)+kr}$ is at least $(d(r+1)+kr)/(nr+d)$, which tends to $(d+k)/n$ as $r \to \infty$. From claim 3.23 we can choose $d = \lfloor \log \binom{n}{k} \rfloor$. If we substitute k with xn, then a short calculation shows that to maximize $(d+k)/n$ we have to maximize $x + H(x)$, where $H(x) = -x \log x - (1-x)\log(1-x)$ (entropy). By derivation we can determine that this function attains its maximum at $x = \frac{2}{3}$, giving a value of $\log 3$ as a lower bound for the limit. □

4 Discussion and open problems

In the literature of conflict-free coloring, hypergraphs that are induced by geometric shapes have been in the focus. It would be interesting to show possible relations of the respective chromatic numbers in this setting.

The exact relationship between the two chromatic numbers with respect to paths for general graphs still remains an open problem. In [4], only graphs which have unique-maximum chromatic number about twice the conflict-free chromatic number where exhibited, but the only bound proved on $\chi^{\mathrm{p}}_{\mathrm{um}}(G)$ was exponential in $\chi^{\mathrm{p}}_{\mathrm{cf}}(G)$. In fact it is even possible that $\chi^{\mathrm{p}}_{\mathrm{um}}(G) \leq 2\chi^{\mathrm{p}}_{\mathrm{cf}}(G) - 2$. The first step to prove this would be to show that $\chi^{\mathrm{p}}_{\mathrm{um}}(T) = O(\chi^{\mathrm{p}}_{\mathrm{cf}}(T))$ for trees. It would also be interesting to extend our results to other classes of graphs.

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Appendix

4.1 The proof of theorem 2.2

We abbreviate conflict-free coloring as cf coloring and unique-max coloring as um coloring.

Proof. First we show an algorithm that achieves a um coloring with $n - \lceil n/k \rceil - l + 4$ colors. Given an l-uniform hypergraph H on n vertices with $\chi_{\text{cf}}(H) = k$, take a cf coloring of H using k colors. Color the vertices of the biggest color class P_1 with color 1, color $\min(l-2, |P_2|)$ vertices of the second biggest color class P_2 with color 2, finally color all other vertices with all different colors. We prove that this is a um coloring of H, i.e. every edge of H is um colored. Take an arbitrary edge e, denote by c the biggest color among its vertices' colors. If c is bigger than 2, then the edge is um colored as only one vertex has color c. If c is 2 then e can be covered by P_1 and P_2. By the definition of the partition, every edge intersects some P_i in exactly one vertex (as the original coloring was conflict-free). Thus, e intersects P_1 or P_2 in exactly one vertex (and the other in $l-1$). If e intersects P_2 in $l-1$ vertices, then it covers a vertex with color bigger than 2, a contradiction. If e intersects P_2 in 1 vertex and P_1 in $l-1$ then it is um colored.

The number of colors x used in this coloring is $1+1+(|P_2| - \min(l-2, |P_2|)) + (n - |P_1| - |P_2|) = n - |P_1| - \min(l-2, |P_2|) + 2$. We finish the proof of the upper bound by showing that this is at most $n - \lceil n/k \rceil - l + 4$. If $|P_2| \geq l-2$ then this number is exactly $n - \lceil n/k \rceil - l + 4$ indeed. Otherwise $|P_2| = l'$ for some $l' < l-2$. If $|P_1| \geq \lceil n/k \rceil + l - l' - 2$ then we are done. If $|P_1| < \lceil n/k \rceil + l - l' - 2$ then using the fact that $n \leq |P_1| + (k-1)|P_2|$, easy computation shows that $n < kl$, contradicting our assumption.

For a given k and l and $n \geq 2kl$ equality holds for the l-uniform hypergraph H defined in the following way. We have n vertices partitioned into k almost equal parts $P_1, P_2, \ldots, P_k$, the first k' having size $\lceil n/k \rceil$, the rest having size $\lceil n/k \rceil - 1$. The edges of the graph are all the edges of size l for which there is a part P_i that intersects the edge in exactly one vertex. During the rest of the proof we will use several times the pigeonhole principle on the above defined parts.

Trivially, a coloring defined by the partition is a cf coloring using k colors, yet we also have to prove that there is no cf coloring using less than k of colors. For that, take a cf coloring of H with the least possible number of colors.

For a color c, if its color class C is covered by some P_i then its size is at most $|P_i|$. Thus, we have at most k' such color classes of size $\lceil n/k \rceil$ and the rest is of size at most $\lceil n/k \rceil - 1$. For a color c for which its color class C is not covered by one part of the partition, C must intersect at least two different parts, P_i and $P_j \neq P_i$. If $|C| > 2l-4$ then either there is an l-subset of C having exactly one point from P_i or there is an l-subset of C having exactly one point from P_j, which is a contradiction as these l-subsets would be monochromatic edges of H. If $|C| \leq 2l-4$ then using our assumption $n \geq 2kl$ implies $2l - 4 < \lceil n/k \rceil - 1$, i.e. if a color class is not covered by some P_i then it is smaller than $\lceil n/k \rceil - 1$. These imply that the only way to color all the vertices using only k colors is that if we do not have color classes intersecting two parts and every color class equals to a part of the partition. By this we proved that if $n \geq 2kl$ then $\chi_{\text{cf}}(H) = k$ and also that the only optimal coloring is the one defined by the partition.

Now take a um coloring with the least possible number of colors, we have to prove that it uses at least $n - \lceil n/k \rceil - l + 4$ colors. We define c to be the *biggest color* for which there are at least 2 vertices having color c. By definition every color bigger than c is used only at most once in this um coloring. We define C as the union of the vertices with color c and C' as the union of the vertices having color c or smaller.

Observation 4.1. *The um coloring uses $n - |C'| + c$ colors.*

As every edge is um colored, the following is true.

Observation 4.2. *There is no edge that contains only vertices from C' and contains at least two vertices from C.*

If C' can be covered by some P_i, then $|C'| \leq \lceil n/k \rceil$ and so we used at least $n - \lceil n/k \rceil + 1 \geq n - \lceil n/k \rceil - l + 4$ colors altogether. If C' cannot be covered by one P_i of the partition, then we have 3 cases:

(i) C cannot be covered by one part.

In this case there are two vertices with color c that are in different parts, x in P_i and y in some $P_j \neq P_i$. If $|C'| > 2l - 4$, then there is an l-subset of C' containing only x from P_i and $l - 1$ vertices from other parts (including y) or an l-subset of C' containing only y from P_j and $l - 1$ vertices from other parts (including x). Any of these two subsets would be an edge of H contradicting observation 4.2. Thus $|C'| \leq 2l - 4$ and so we used at least $n - (2l - 4) + 1 \geq n - \lceil n/k \rceil - l + 4$ colors (for the last inequality we used that $n \geq 2kl$).

(ii) C is contained in some P_i and C' can be covered by two parts P_i and P_j.

If $|C' \cap P_i| > l - 2$ then there exists an l-subset of C' containing exactly one vertex from P_j and $l - 1$ vertices from P_i such that at least two of these vertices have color c. This subset would be an edge of H contradicting observation 4.2. Thus, $|C'| \leq |P_j| + l - 2 \leq \lceil n/k \rceil + l - 2$, and as $c \geq 2$ in this case, we used at least $n - (\lceil n/k \rceil + l - 2) + 2 = n - \lceil n/k \rceil - l + 4$ colors.

(iii) C is contained in some P_i and C' cannot be covered by two parts.

In this case C' contains points from at least three parts, P_i, P_j and some P_h. Now it is easy to see that if $|C'| > 2l - 6$ then there is an l-subset of C' containing at least two vertices from P_i that either has exactly one vertex from P_j or exactly one vertex from P_h. This subset would be an edge in H contradicting observation 4.2. Thus, $|C'| \leq 2l - 6$, and as $c \geq 2$, we used at least $n - (2l - 6) + 2 \geq n - \lceil n/k \rceil - l + 4$ colors. □