Choosability in geometric hypergraphs*

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Abstract

Given a hypergraph $H = (V, \mathcal{E})$, a coloring of its vertices is said to be conflict-free if for every hyperedge $S \in \mathcal{E}$ there is at least one vertex in S whose color is distinct from the colors of all other vertices in S. The study of this notion is motivated by frequency assignment problems in wireless networks. We introduce and study the list-coloring (or choice) version of this notion. List coloring arises naturally in the context of wireless networks.

1 Introduction and preliminaries

Before introducing our results, let us start with several definitions and notations that will be used throughout the paper.

Definition 1.1. Let $H = (V, \mathcal{E})$ be a hypergraph and let C be a coloring $C: V \to \mathbb{N}^+$:

- We say that C is a *proper coloring* if for every hyperedge $S \in \mathcal{E}$ with $|S| \geq 2$ there exist two vertices $u, v \in S$ such that $C(u) \neq C(v)$. That is, every hyperedge with at least two vertices is non-monochromatic.
- We say that C is a *conflict-free coloring* (cf-coloring in short) if for every hyperedge $S \in \mathcal{E}$ there exists a color $i \in \mathbb{N}$ such that $|S \cap C^{-1}(i)| = 1$. That is, every hyperedge $S \in \mathcal{E}$ contains some vertex whose color is unique in S.

We denote by $\chi(H)$ the minimum integer k for which H admits a proper coloring with a total of k colors. We denote by $\chi_{\mathrm{cf}}(H)$ the minimum integer k for which H admits a cf-coloring with a total of k colors. Obviously, every cf-coloring of H is also a proper coloring but the converse is not necessarily true. Thus, we have: $\chi_{\mathrm{cf}}(H) \geq \chi(H)$.

For several geometric hypergraphs, the parameters $\chi_{\mathrm{cf}}(H)$ and $\chi(H)$ are well studied and in some cases well understood. The study of cf-coloring was initiated in the work of Even et al. [18] and of Smorodinsky [28] and was extended by numerous other papers (c.f., [1, 4, 9, 12, 13, 14, 19, 23, 26, 27]). The study was initially motivated by its application to frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile clients*. Base stations have fixed positions and provide the backbone of the network; they can be modeled, say, as discs in the plane that represent the area covered by each base station's antenna. Every base station emits at a fixed frequency. If a client wants to establish a link with a base

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station it has to tune itself to this base station's frequency. Clients, however, can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any point p in the plane (representing location of a client), there must be at least one base station which covers p and with a frequency that is not used by any other base station covering p. Since frequencies are limited and costly, a scheme that reuses frequencies, where possible, is desirable. Let us formulate this in the language of hypergraph coloring. Let D be the set of discs representing the antennas. We thus seek the minimum number of colors k such that one can assign each disc with one of the k colors so that in every point p in the union of the discs in D, there is at least one disc $d \in D$ that covers p and whose color is distinct from all the colors of other discs containing p. This is equivalent to finding the cf-chromatic number of a certain hypergraph $H = H(D)$ whose vertex set is D and whose hyperedges are defined by the Venn diagram of D. Below, we give a formal definition for $H(D)$.

Until now, research on cf-coloring was mainly focused on bounds on the cf-chromatic number of the underlying hypergraphs. In real life, it make sense to assume that each antenna in the wireless network is further restricted to use a subset of the available spectrum. This restriction might be local (depending, say, on the physical location of the antenna). Hence, different antennas might have different subsets of frequencies available for them. Thus, it makes sense to study the list version of conflict-free coloring. That is, assume further that each antenna $d \in D$ is associated with a subset L_d of frequencies. We want to assign to each antenna d a frequency that is taken from its allowed set L_d. The following problem thus arises. What is the minimum number $f = f(n)$ such that given any set D of n antennas (represented as discs) and any family of subsets of integers $\mathcal{L} = \{L_d\}_{d \in D}$ associated with the antennas in D, the following holds: If each subset L_d is of cardinality f, then one can cf-color the hypergraph $H = H(D)$ from $\mathcal{L}$. In what follows, we give a formal definition of the coloring model.

Definition 1.2. Let $H = (V, \mathcal{E})$ be a hypergraph and let $\mathcal{L} = \{L_v\}_{v \in V}$ be a family of $|V|$ subsets of positive integers. We say that H admits a cf-coloring from $\mathcal{L}$ (respectively, a proper coloring from $\mathcal{L}$) if there exists a cf-coloring (respectively a proper coloring) $C\colon V \to \mathbb{N}^+$ such that $C(v) \in L_v$ for every $v \in V$.

Definition 1.3. We say that a hypergraph $H = (V, \mathcal{E})$ is *k-cf-choosable* (respectively, *k-choosable*) if for every family $\mathcal{L} = \{L_v\}_{v \in V}$ such that $|L_v| \geq k \ \forall v \in V$, H admits a cf-coloring (respectively a proper coloring) from $\mathcal{L}$.

In this paper we are interested in the minimum number k for which a given hypergraph is k-cf-choosable (respectively, k-choosable). We refer to this number as the *cf-choice number* (respectively the *choice number*) of H and denote it by $ch_{\rm cf}(H)$ (respectively $ch(H)$). Obviously, if the cf-choice number (respectively, the choice number) of H is k then it can be cf-colored (respectively properly colored) with at most k colors, as one can cf-color (respectively, properly color) H from $\mathcal{L} = \{L_v\}_{v \in V}$ where for every v we have $L_v = \{1, \ldots, k\}$. Thus,

$$ch(H) \geq \chi(H) \tag{1}$$

and

$$ch_{\rm cf}(H) \geq \chi_{\rm cf}(H). \tag{2}$$

Hence, any lower bound on $\chi_{\rm cf}(H)$ is also a lower bound on $ch_{\rm cf}(H)$.

The study of list coloring for the special case of graphs, i.e., 2-uniform hypergraphs, was initiated in [17, 33]. List proper coloring of hypergraphs has been studied more recently, as well; see, e.g., [22]. We refer the reader to the survey of Alon [3] for more on list coloring of graphs.

In this paper we initiate the study of cf-choice number of hypergraphs. We provide bounds on the cf-choice number and also on the choice number of various hypergraphs. We focus mainly on geometric hypergraphs. Our main result is an asymptotically tight bound of $O(\log n)$ on the cf-choice number of $H(D)$ when D is a family of n unit-discs. We also study the choice number of several geometric hypergraphs and show that many of the known bounds for the proper coloring of the underlying hypergraphs hold in the context of list-coloring as well. We also provide an asymptotically tight upper bound on the cf-choice number of an arbitrary hypergraph in terms of its cf-chromatic number and the number of its vertices.

Geometric hypergraphs: Let P be a set of n points in the plane and let $\mathcal{R}$ be a family of regions in the plane (such as all discs, all axis-parallel rectangles, etc.). We denote by $H = H_{\mathcal{R}}(P)$ the hypergraph on the set P whose hyperedges are all subsets P' that can be cut off from P by a region in $\mathcal{R}$. That is, all subsets P' such that there exists some region $r \in \mathcal{R}$ with $r \cap P = P'$. We refer to such a hypergraph as the hypergraph *induced by P with respect to $\mathcal{R}$*.

For a finite family $\mathcal{R}$ of planar regions, we denote by $H(\mathcal{R})$ the hypergraph whose vertex set is $\mathcal{R}$ and whose hyperedge set is the family $\{\mathcal{R}_p \mid p \in \mathbb{R}^2\}$ where $\mathcal{R}_P \subset \mathcal{R}$ is the subset of all regions in $\mathcal{R}$ that contain p. We refer to such a hypergraph as the hypergraph *induced by* $\mathcal{R}$. Informally, this is the Venn-diagram of the family $\mathcal{R}$.

Consider for example the (infinite) family D of all discs in the plane. The following natural question arises: What is the minimum number $f = f(n)$ such that for any finite set P of n points we have $ch_{\text{cf}}(H_D(P)) \le f(n)$. Similar questions can be asked for other families of geometric hypergraphs where one is interested in both bounds on the choice-number and the cf-choice number of such hypergraphs.

In section 2 we provide near-optimal upper bounds on the (non-monochromatic) choice number of several geometric hypergraphs. In section 3 we focus on a generalization of the so-called discrete intervals hypergraph. Geometrically, this hypergraph can be described as being induced by a set of n points in $\mathbb{R}^r$ with respect to all axis-parallel strips in $\mathbb{R}^r$, where an axis-parallel strip in $\mathbb{R}^r$ is the region enclosed between two parallel hyperplanes which are also parallel to one of the r axes. We prove that the cf-choice number of such a hypergraph is at most $c(r)\log n$, where $c(r)$ is a constant that depends only on the dimension r. We apply this result and combine it with several more geometric and probabilistic ideas to obtain an asymptotically tight upper bound on the cf-choice number of a hypergraph induced by points in the plane with respect to the family of all unit-discs. Our bound on the cf-choice number of hypergraphs induced by points with respect to unit-discs is detailed in Section 4. Most of our proofs combine geometric and probabilistic ideas. We also provide a deterministic proof of the weak upper bound $ch_{\text{cf}}(H) = O(\sqrt{n})$ when H is a hypergraph induced by n points in the plane with respect to arbitrary discs. Even though, for such a hypergraph H, we obtain a significantly improved bound on $ch_{\text{cf}}(H)$ of $O(\log^2 n)$, later in Section 5, we believe that the proof of the weak bound is of independent interest. In particular, using similar arguments we obtain an asymptotically tight bound on the cf-choice number of hypergraphs consisting of the vertices of a planar graph together with all subsets of vertices that form a simple path (see [11] for applications of this class of hypergraphs).

Finally, in Section 5 we provide a general bound on the cf-choice number of any hypergraph in terms of its cf-chromatic number. We show that for any hypergraph H (not necessarily of a geometric nature) with n vertices we have: $ch_{\text{cf}}(H) \le \chi_{\text{cf}}(H) \cdot \ln n + 1$. The proof of this fact uses a simple probabilistic argument, which is an extension of a probabilistic argument first given in [17]. There, it was proved that the choice-number of every bipartite graph with n vertices is at most $\log n + 1$. As a corollary, we obtain the bound $ch_{\text{cf}}(H) = O(\log^2 n)$ for any hypergraph H induced by n points in the plane with respect to arbitrary discs. Table 1 summarizes our results, together

Hypergraph	$\chi(H)$	$ch(H)$	$\chi_{\text{cf}}(H)$	$ch_{\text{cf}}(H)$
intervals in r permutations on $\{1,\ldots,n\}$	$2r$	$2r$	$\Theta(\log n)$	$\Theta(\log n)$
points w.r.t unit discs	4 [29]	5	$\Theta(\log n)$ [18]	$\Theta(\log n)$
n regions with linear union complexity	$O(1)$ [29]	$O(1)$	$\Theta(\log n)$ [19, 29]	$O(\log^2 n)$
paths in planar graphs	4 [7, 8]	5 [32]	$\Theta(\sqrt{n})$ [20, 11]	$\Theta(\sqrt{n})$
arbitrary hypergraph	—	$\leq \chi(H)\ln n+1$	—	$\leq \chi_{\text{cf}}(H)\ln n+1$

Table 1: Summary of results

with related results from other papers (which are referenced in the table).

2 Choice number of geometric hypergraphs

In this section we provide near-optimal upper bounds on the choice number of several geometric hypergraphs. We need the following definitions:

Definition 2.1. Let $\mathcal{R}$ be a family of n simple Jordan regions in the plane. The *union complexity* of $\mathcal{R}$ is the number of vertices (i.e., intersection of boundaries of pairs of regions in $\mathcal{R}$) that lie on the boundary $\partial\bigcup_{r\in\mathcal{R}} r$.

Definition 2.2. Let $H=(V,\mathcal{E})$ be a hypergraph. Let $G=(V,E)$ be the graph whose edges are all hyperedges of $\mathcal{E}$ with cardinality two. We refer to G as the *Delaunay graph* of H.

Theorem 2.3. *(i) Let H be a hypergraph induced by a finite set of points in the plane with respect to discs. Then $ch(H)\leq 5$.*

(ii) Let D be a finite family of discs in the plane. Then $ch(H(D))\leq 5$.

(iii) Let $\mathcal{R}$ be a set of n regions and let $\mathcal{U}:\mathbb{N}\to\mathbb{N}$ be a function such that $\mathcal{U}(m)$ is the maximum complexity of any k regions in $\mathcal{R}$ over all $k\leq m$, for $1\leq m\leq n$. We assume that $\frac{\mathcal{U}(m)}{m}$ is a non-decreasing function. Then, $ch(H(\mathcal{R}))=O(\frac{\mathcal{U}(n)}{n})$.

Proof. (i) Consider the Delaunay graph $G=G(P)$ on P, where two points p and q form an edge in G if and only if there exists a disc d such that $d\cap P=\{p,q\}$. That is, there exists a disc d that cuts off p and q from P. The proof of (i) follows easily from the following known facts:

1. Every disc containing at least two points of P must also contain a Delaunay edge $\{p,q\}\in E(G)$. (see, e.g., [18]).
2. G is planar (see, e.g., [15]).
3. Every planar-graph is 5-choosable [32].

(ii) The proof of the second part follows from a reduction to three dimensions from [29] and Thomassen's result [32].

(iii) For the third part of the theorem, we need the following lemma from [29]:

Lemma 2.4. *[29] Let $\mathcal{R}$ be a set of n regions and let $\mathcal{U}:\mathbb{N}\to\mathbb{N}$ be a function such that $\mathcal{U}(m)$ is the maximum complexity of any k regions in $\mathcal{R}$ over all $k\leq m$, for $1\leq m\leq n$. Then, the Delaunay graph G of the hypergraph $H=H(\mathcal{R})$ has a vertex with degree at most $c\frac{\mathcal{U}(n)}{n}$ where c is some absolute constant.*

The proof is similar to the proof of [29] of the fact that $\chi(H(\mathcal{R})) = O(\frac{\mathcal{U}(n)}{n})$. We prove that $ch(H(\mathcal{R})) \leq c \cdot \frac{\mathcal{U}(n)}{n} + 1$. Let $\mathcal{L} = \{L_r\}_{r\in\mathcal{R}}$ be the sets associated with the regions of $\mathcal{R}$. The proof is by induction on n. Let $r \in \mathcal{R}$ be a region with at most $c \cdot \frac{\mathcal{U}(n)}{n}$ neighbors in G. By the induction hypothesis, the hypergraph $H(\mathcal{R} \setminus \{r\})$ is $c \cdot \frac{\mathcal{U}(n-1)}{n-1} + 1 \leq c \cdot \frac{\mathcal{U}(n)}{n} + 1$-choosable (by our monotonicity assumption on $\frac{\mathcal{U}(n)}{n}$). We need to choose a color (out of the $c \cdot \frac{\mathcal{U}(n)}{n} + 1$ colors that are available for us in the set L_r) for r such that the coloring of $\mathcal{R}$ is valid. Obviously, points that are not covered by r are not affected by the coloring of r. Note also that any point $p \in r$ that is contained in at least two regions of $\mathcal{R} \setminus r$ is not affected by the color of r since, by induction, the set of regions in $\mathcal{R} \setminus \{r\}$ containing such points is non-monochromatic. We thus only need to color r with a color that is different from the colors of all regions $r' \in \mathcal{R} \setminus r$, for which there is a point p that is contained only in $r \cap r'$. However, by our choice of r, there are at most $c \cdot \frac{\mathcal{U}(n)}{n}$ such regions. Thus, we can assign to r a color among the $c \cdot \frac{\mathcal{U}(n)}{n} + 1$ colors available to us in L_r and keep the coloring of $\mathcal{R}$ proper. This completes the inductive step. □

Corollary 2.5. *Let $\mathcal{P}$ be a family of n pseudo-discs (i.e., a family of simple closed Jordan regions, such that the boundaries of any two of them intersect at most twice). Then $ch(H(\mathcal{P})) = O(1)$.*

The corollary follows immediately from the fact that such a family $\mathcal{P}$ has linear union complexity [21], combined with Theorem 2.3.

3 Permutations hypergraphs

Let $[n] = \{1, \ldots, n\}$. For $s \leq t$, $s, t \in [n]$, we define the (discrete) interval $[s, t] = \{i \mid s \leq i \leq t\}$. The *discrete interval hypergraph* H_n has vertex set $[n]$ and hyperedge set $\{[s, t] \mid s \leq t,\ s, t \in [n]\}$. It is not difficult to prove that $\chi_{\text{cf}}(H_n) = \lfloor \log_2 n \rfloor + 1$ (see, e.g., [18, 28]). Therefore, from inequality (2), we have the lower bound $ch_{\text{cf}}(H_n) \geq \lfloor \log_2 n \rfloor + 1$. Hence, the following upper bound is tight:

Proposition 3.1. *For every $n \geq 1$, $ch_{cf}(H_n) \leq \lfloor \log_2 n \rfloor + 1$.*

Proof. Assume, without loss of generality, that $n = 2^k + 1$. We will show that H_n is $k + 1$ cf-choosable. The proof is by induction on k. Let $\mathcal{L} = \{L_i\}_{i\in[n]}$, such that $|L_i| = k + 1$, for every i. Consider the median vertex $p = 2^{k-1} + 1$. Choose a color $x \in L_p$ and assign it to p. Remove x from all other lists (for lists containing x), i.e., consider $\mathcal{L}' = \{L'_i\}_{i\in[n]\setminus p}$ where $L'_i = L_i \setminus \{x\}$. Note that all lists in $\mathcal{L}'$ have size at least k. The induction hypothesis is that we can cf-color any set of points of size $2^{k-1} + 1$ from lists of size k. Thus, we cf-color vertices smaller than p and independently vertices larger than p, both using colors from the lists of $\mathcal{L}'$. However, we need to argue that intervals that contain the median vertex p also contain some unique color. But this is obviously true because p itself is colored uniquely (with color x) in such intervals. This completes the induction step and hence the proof of the proposition. □

For a permutation $\pi\colon [n] \to [n]$, we define the $[s, t]$-*strip* as the set $\pi_{[s,t]} = \{\pi(i) \mid s \leq i \leq t\}$. Then, the *strip hypergraph* of π is $H_\pi = ([n], \mathcal{E}(H_\pi))$, where $\mathcal{E}(H_\pi) = \{\pi_{[s,t]} \mid s \leq t,\ s, t \in [n]\}$. It is not difficult to see that H_π is isomorphic to H_n.

We now consider, for r permutations on the same n elements, the union of the corresponding r strip hypergraphs. Namely, for r permutations on n elements, $\pi^1, \ldots, \pi^r$, we consider the hypergraph $H_{\pi^1,\ldots,\pi^r} = ([n], \mathcal{E}(H_{\pi^1}) \cup \cdots \cup \mathcal{E}(H_{\pi^r}))$. We will prove that $ch_{\text{cf}}(H_{\pi^1,\ldots,\pi^r}) \leq c(r) \log n$ and this upper bound will be useful when we study the cf-choice number of the hypergraph induced by points in the plane with respect to unit discs.

We note that the bound $\chi_{\mathrm{cf}}(H_{\pi^1,\ldots,\pi^r}) = O(\log n)$ follows by a general framework of [29] and the fact that the Delaunay graph $G(H_{\pi^1,\ldots,\pi^r})$ is $2r$ colorable (see the end of this section).

Remark 3.2. Permutations hypergraphs were studied recently in the context of combinatorial discrepancy as well (see, e.g., [10, 30]) and a major problem of determining the discrepancy of such hypergraphs is still widely open.

The notion of *induced subhypergraph* will be needed, so we define it here: Given a hypergraph $H = (V, \mathcal{E})$ and a subset $V' \subset V$, put $H[V'] = (V', \{e \cap V' \mid e \in \mathcal{E},\, e \cap V' \neq \emptyset\})$. We refer to $H[V']$ as the *subhypergraph of H induced by V'*.

Theorem 3.3. *There exists a constant $c = c(r)$ such that for any r permutations $\pi^1, \ldots, \pi^r \colon [n] \to [n]$, $ch_{cf}(H_{\pi^1,\ldots,\pi^r}) \leq c \log n$.*

Proof. Put $H = H_{\pi^1,\ldots,\pi^r}$.

By a recent result of Aloupis et al. [6], there is a coloring $C_* \colon [n] \to [r]$ of H such that every hyperedge e of H, with cardinality at least $q_r = 5r \ln r$, is colorful. That is, for every $i \in [r]$, there exists a vertex $v \in e$ with $C_*(v) = i$. In [6], such a coloring is referred to as a *polychromatic coloring*, and can be computed efficiently.

We construct the following graph G on $V = [n]$: add an edge xy to $E(G)$ if $\{x, y\} \subset e$ for some hyperedge $e \in \mathcal{E}(H)$ with $|e| < q_r$. The hyperedges $\mathcal{E}(H_{\pi^i})$ of each permutation π^i, for a fixed $v \in V$ contribute at most $2(q_r - 1)$ neighbors to v in G. Thus, the maximum degree Δ of G is bounded by $2(q_r - 1)r$, and thus G can be greedily colored with at most $2(q_r - 1)r + 1$ colors. Denote such a coloring by C_G.

We assign the following 'type' to each $v \in V$: $T(v) = (C_*(v), C_G(v))$. The number K of distinct types is bounded by $(2(q_r - 1)r + 1)r$.

Consider a family $\mathcal{L} = \{L_v\}_{v \in V}$, such that for every v, $|L_v| = c \ln n$, where c is a constant to be determined later, depending only on r. We wish to find a family $\mathcal{L}' = \{L'_v\}_{v \in V}$ with the following properties:

1. $\forall v \in V$, $L'_v \subset L_v$.
2. $\forall v \in V$, $|L'_v| \geq 1 + \log_2 n$.
3. For $v, u \in V$, if $T(v) \neq T(u)$, then $L'_v \cap L'_u = \emptyset$.

We claim that, if such a family $\mathcal{L}'$ exists, then there exists a cf-coloring of H from $\mathcal{L}'$: For every $i \in [r]$, consider the vertices colored with color i in the colorful coloring C_*, i.e., $V^i = C_*^{-1}(i)$. It is not difficult to see that for any permutation $\pi \colon [n] \to [n]$ and for any $S \subset [n]$, the subhypergraph induced by S, $H_\pi[S]$, is isomorphic to the discrete interval hypergraph $H_{|S|}$. Thus, the hypergraph $H_{\pi^i}[V^i]$ is a discrete interval hypergraph with at most n vertices, and by proposition 3.1 we can cf-color it from lists of size $1 + \lfloor \log_2 n \rfloor$. We apply this coloring process for each of $H_{\pi^i}[V^i]$, for $i \in [r]$. Next, we prove that this is, indeed, a cf-coloring of the hypergraph H: Consider a hyperedge $e \in \mathcal{E}(H_{\pi^i})$. If $e \cap V^i \neq \emptyset$, then $e \cap V^i$ is a discrete interval in $H_{\pi^i}[V^i]$ and thus it contains a vertex v with unique color among vertices of $e \cap V^i$. Since the type of no vertex in $e \cap V^i$ is the same as the type of a vertex in $e \setminus V^i$, the color of vertex v is uniquely occurring in e. If $e \cap V^i = \emptyset$, then by the property of C_*, $|e| < q_r$ and, therefore, every pair of vertices in e have distinct types. Hence, by property 3 of the family $\mathcal{L}'$ the lists of $\{L'_v\}_{v \in e}$ are pair-wise disjoint and thus e has the cf property (in fact it has the stronger "rainbow" property, i.e., every $v \in e$ gets a uniquely occurring color in e).

Thus, it is left to prove that such a family $\mathcal{L}'$ do exists. The proof is probabilistic. For each element in $\cup\mathcal{L}$, assign it uniformly at random to one of the K distinct types. For every $v \in V$, keep a color i of L_v in the updated list L'_v if and only if i was assigned to type $T(v)$. Properties 1 and 3 are trivially satisfied by $\mathcal{L}'$.

For a fixed $v \in V$, let X_v denote the number of colors in the list L_v that are assigned the type $T(v)$, and thus belong to the list L'_v. X_v is a binomial random-variable with success probability $p = 1/K$ and expectation $\mu = p|L_v|$. By the Chernoff bound (see, e.g., [5]) we have:

$$\Pr[X_v < (1-\delta)\mu] < \exp(-\mu\delta^2/2)$$

Choosing δ so that $(1-\delta)\mu = 2\ln n/\ln 2 = 2\log_2 n \geq 1 + \lfloor \log_2 n \rfloor$ (for $n > 1$), we get that the probability of the "bad" event that the vertex v will not have a sufficient number of colors in L'_v is

$$\Pr[X_v < 1 + \lfloor \log_2 n \rfloor] < \exp(-(c\ln n)(1 - (2K/(c\ln 2)))^2/2).$$

By the union bound, the probability that at least one of the n vertices does not have sufficient many colors in its list is at most

$$n\exp(-(c\ln n)(1-(2K/(c\ln 2)))^2) = n^{1-c(1-(2K/(c\ln 2)))^2/2},$$

which is less than 1 for $c \geq 2K(1 + 2/\ln 2)$ and $n > 1$. Therefore, property 2 is also satisfied with a positive probability. Hence, such a family $\mathcal{L}'$ exists. This completes the proof of the theorem. □

We also give tight bounds of $2r$ for the (non-monochromatic) chromatic and choice number of a hypergraph of r permutations.

Proposition 3.4. *For any r permutations $\pi^1, \ldots, \pi^r \colon [n] \to [n]$, $ch(H_{\pi^1,\ldots,\pi^r}) \leq 2r$.*

Proof. Set $H = H_{\pi^1,\ldots,\pi^r}$ and assume that for every $v \in [n]$, $|L_v| = 2r$. For each permutation π^i, define the path P^{π^i} on n vertices with edge set

$$E(P^{\pi^i}) = \{\{\pi^i_j, \pi^i_{j+1}\} \mid 1 \leq j < n\}.$$

It is not difficult to see that the Delaunay graph $G(H)$ is the union of the r paths $P^{\pi^1}, \ldots, P^{\pi^r}$. Moreover, a (proper) coloring of $G(H)$ is also a non-monochromatic coloring of H. Consider the vertices of $G(H)$ in the order of one of the permutations, say π^1. We color $G(H)$ as follows: Iteratively, following the above order, we color each vertex v greedily with the smallest color in L_v which is not used by any of the already colored neighbors of v. At the end, the above method produces a proper coloring of $G(H)$ as long as there is an available color for every vertex. Vertex v has at most one already colored vertex neighboring on path P^{π^1} and at most $2(r-1)$ vertices neighboring on the other $r-1$ paths, i.e., a total of at most $2r-1$ already colored neighbors. Therefore, since $|L_v| = 2r$, the above method colors all vertices. □

The above result is tight for both the chromatic and choice number, because of inequality (1) and the following proposition:

Proposition 3.5. *For every r, there exist permutations $\pi^1, \ldots, \pi^r$, such that $\chi(H_{\pi^1,\ldots,\pi^r}) \geq 2r$.*

Proof. It is known from [31] that the edges of K_{2r} can be covered by r pairwise edge-disjoint Hamiltonian paths, $P^1, \ldots, P^r$. If $V(K_{2r}) = [2r]$ then each path P^i can be seen as a permutation $\pi^i \colon [2r] \to [2r]$. Consider the hypergraph $H = H_{\pi^1,\ldots,\pi^r}$. A non-monochromatic coloring of H is also a coloring of its Delaunay graph $G(H)$, i.e., $\chi(H) \geq \chi(G(H))$. But $G(H) = K_{2r}$ and thus $\chi(H) \geq 2r$. □

4 CF-choice number of Geometric Hypergraphs

In this section we provide upper bounds on the cf-choice number of hypergraphs induced by a finite set of points in the plane with respect to ranges such as half-planes, unit-discs and arbitrary discs.

We start with the case of a hypergraph H induced by n points in the plane with respect to half-planes. If the points of P are in convex position, we face a case which is almost identical to the discrete interval hypergraph (see proposition 3.1). We only need to modify the first stage of the recursion where we take two antipodal points along their order on the convex hull and break the problem into two sub-problems of size $\frac{n-2}{2}$. Thus, if the lists attached to the points are of size at least $\log n + 2$ then we can find a cf-coloring from the lists. A difficulty arises when there are many interior points, for example, when $n/2$ points lie on the boundary of $\mathrm{CH}(P)$ and $n/2$ points lie in the interior of $\mathrm{CH}(P)$. In the standard cf-coloring (without lists) we could simply assign to all of the interior points one color and never use it again. However, in our case, we must assign the colors from the corresponding lists. Those lists might be pairwise disjoint which means that we assigned $n/2$ distinct colors to the interior points. If we want the property that those colors are never used for the extreme points, it might be the case that we need to remove, from each of the lists of the extreme points, $n/2$ colors. Then the lists should be of size at least $n/2$. To overcome this difficulty, we apply again the probabilistic method.

Theorem 4.1. *Let P be a set of n points in the plane. Let $H = H(P)$ be the hypergraph induced by P with respect to all half-planes. Then $ch_{cf}(H) \leq c \log n$.*

Proof. Let $\mathcal{L} = \{L_p\}_{p \in P}$ be a family of n lists attached to points of P. Assume that all lists in $\mathcal{L}$ have cardinality at least $c \log n$, where c is some absolute constant to be determined later. We need to show that P admits a cf-coloring from $\mathcal{L}$. Let P_1 be the extreme points of P (i.e., the points that belong to the boundary of the convex-hull $\mathrm{CH}(P)$). Let $P_2 = P \setminus P_1$ be the interior points of $\mathrm{CH}(P)$. Next, we modify the lists of $\mathcal{L}$ by taking for each point p a subset $L'_p \subset L_p$, obtaining a family of lists $\mathcal{L}'$ with the following properties:

1. $|L'_p| \geq 1 \ \forall p \in P_2$. Namely, sub-lists taken for P_2 are non-empty.

2. $|L'_p| \geq \log n + 2 \ \forall p \in P_1$. Namely, the lists taken for the extreme points are "large enough".

3. For every pair p, q such that $p \in P_1, q \in P_2$, we have $L'_p \cap L'_q = \emptyset$. That is, every color x left in any list of the interior points, does not belong to any of the lists left for the extreme points.

As before. we do this modification randomly. Let $S = \bigcup_{p \in P} L_p$. Independently for every color $x \in S$ we do the following random choice: With probability $\frac{1}{2}$ we erase x from all lists of points in P_1 and with probability $\frac{1}{2}$ we erase it from all lists of points in P_2. For every $p \in P$, let L'_p denote the list obtained from L_p after the above random procedure. For $p \in P_1$, let A_p denote the "bad" event that $|L'_p| < \log n + 2$ and for $p \in P_2$, let B_p denote the "bad" event that $L'_p = \emptyset$. We need to show that $\mathbf{Pr}\Big[(\bigcup_{p \in P_1} A_p) \cup (\bigcup_{p \in P_2} B_p)\Big] < 1$. Note that for every $p \in P_2$, $\mathbf{Pr}[B_p] \leq 2^{-c \log n} = n^{-c}$ since every list L_p is of size at least $c \log n$ and the probability of a color $x \in L_p$ to be removed is exactly $\frac{1}{2}$. Note also that for every $p \in P_1$ we have:

$$\mathbf{E}\big[|L'_p|\big] = \frac{1}{2}|L_p| \geq \frac{1}{2} c \log n,$$

since every color in L_p is removed with probability $\frac{1}{2}$. We need the following version of the Chernoff inequality (see, e.g., [5]) for a binomial random variable X with k elements and success probability

$\frac{1}{2}$:

$$\mathbf{Pr}[X < \mu - a] \le e^{-\frac{2a^2}{k}}$$

where $\mu = \mathbf{E}[X] = \frac{k}{2}$ and for any a with $0 < a < \mu$.

Put $\mu = \mathbf{E}\big[|L'_p|\big]$ and $a = \mu - (\log n + 2)$. Note that $a \ge \frac{c}{2}\log n - \log n - 2 = \frac{c-2}{2}\log n - 2$. We thus have:

$$\mathbf{Pr}[A_p] = \mathbf{Pr}\big[|L'_p| < \log n + 2\big] \le \mathbf{Pr}\big[|L'_p| < \mu - a\big] \le e^{-\frac{2a^2}{|L_p|}} \le e^{-\frac{2(\frac{c-2}{2}\log n - 2)^2}{c\log n}} \ll 1$$

For any constant c which is sufficiently large. We use the union bound in order to obtain:

$$\mathbf{Pr}\left[(\bigcup_{p\in P_1} A_p) \cup (\bigcup_{p\in P_2} B_p)\right] < |P_1| e^{-\frac{2(\frac{c-2}{2}\log n - 2)^2}{c\log n}} + |P_2| n^{-c} < 1$$

By our choice of c. It is easy to see that for $c \ge 10$ the above inequality holds. Here, we do not attempt to optimize the constant c. Thus, there exists a family $\mathcal{L}'$ such that none of the bad events occur and hence properties 1 and 2 above are satisfied. Note that, by construction, property 3 is trivially satisfied. We choose an arbitrary color $x \in L'_p$ for every point $p \in P_2$ and we cf-color the points of P_1 from the lists $\{L'_p\}_{p\in P_1}$ in the same way as in the one dimensional case (see proposition 3.1). This can be done since the lists $\{L'_p\}_{p\in P_1}$ all have cardinality large enough. Note that this coloring is indeed a valid cf-coloring since for every half-plane h with $h \cap P \neq \emptyset$ we also have $h \cap P_1 \neq \emptyset$ so there is at least one point p whose color $c(p) \in L'_p$ is unique in $h \cap P_1$ and for every color x' used for points in $h \cap P_2$ we must have $x' \neq c(p)$. Thus, $c(p)$ is also unique in $h \cap P$. This completes the proof of the theorem. □

Extending the ideas used in the previous proofs and combining it with a geometric partition and Theorem 3.3, we obtain the main result of our paper:

Theorem 4.2. *Let P be a set of n points in the plane and let $H = H(P)$ denote the hypergraph induced by P with respect to unit-discs. Then $ch_{cf}(H) = O(\log n)$.*

Proof. Let $V = P$ be a set of n points in the plane and consider the family of disks of radius 1 (i.e., unit discs) in the plane. Suppose we are given a family $\mathcal{L} = \{L_v\}_{v\in V}$ such that for every $v \in V$, $|L_v| \ge c\ln n$, where c is a constant to be determined later.

Consider a regular tiling of the plane with squares of side 1/2, such that no point in V intersects with the boundaries of the squares. It is a well known fact that we can partition the squares into a constant number of classes K so that no unit disk contains points in V belonging to two different squares in the same class. Assign to each point $v \in V$ the class of the square that contains it.

We would like to compute a family $\mathcal{L}' = \{L'_v\}_{v\in V}$ with the following properties:

1. For every $v \in V$, $L'_v \subset L_v$.
2. For every $v \in V$, $|L'_v| \ge c' \ln n$.
3. If v, u belong to different classes, then $L'_v \cap L'_u = \emptyset$.

The value of the constant c' will be determined later as well. This family $\mathcal{L}'$ can be constructed randomly in a way which is similar to the one used in the proof of Theorem 3.3. We omit the details.

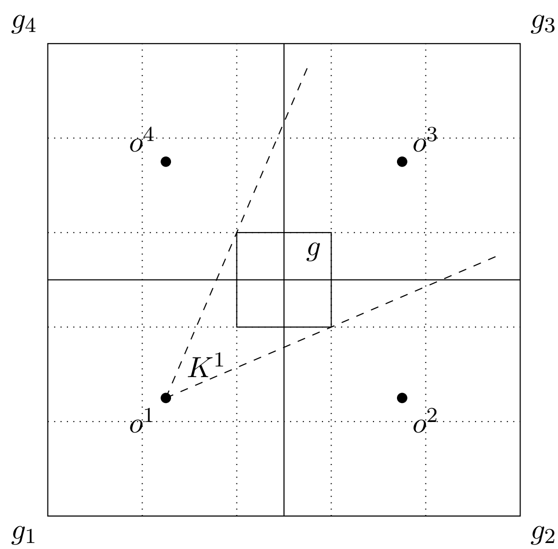

Figure 1: taken from [13]: Squares g, g_1, g_2, g_3, g_4

Obtaining such a family $\mathcal{L}'$ as above, we claim that we can now concentrate in conflict-free coloring points in each square of the grid independently, in order to get a conflict-free coloring of the whole point set V with respect to unit discs.

Hence, it is enough to focus on a fixed cell g of the square tiling. Let D denote the family of unit discs having non-empty intersection with the square g. Next, we partition D into four classes; this partition idea is taken from [13] where it was used for an algorithmic online cf-coloring problem of points in the plane with respect to unit-discs. Consider the square g^0 which is concentric with g, has sides parallel to the sides of g, and has side length $5/2$. Partition g^0 into four squares g^1, g^2, g^3, g^4, each of side length $5/4$, that have sides parallel to the sides of g and share the center of g as a common point. Let o^1, o^2, o^3, o^4 be the centers of g^1, g^2, g^3, g^4, respectively. See Figure 1 for an illustration. It is not difficult to see that the center of a unit disk in D lies in g^0 and, therefore, it belongs to at least one of the squares g^1, g^2, g^3, g^4. Next, we partition D into four sub-families D^1, D^2, D^3, D^4. For every $d \in D$, pick one of the squares that contains its center, say g^i, and put d in D^i. The following lemma is proved in [13]:

Lemma 4.3 ([13])**.** *Let K^i denote the convex cone with apex o^i spanned by g, for $i = 1, \ldots, 4$. Then, for any pair of discs $d, d' \in D^i$, the intersection $\partial d \cap \partial d' \cap K^i$ consists of at most one point.*

In particular, since $g \subset K^i$, for $i = 1, \ldots, 4$, the boundaries of the unit discs belonging to the same class D^i behave like pseudolines inside g.

We set $V_g = g \cap V$. We call a point $v \in V_g$ *i-extreme* if there exists a unit disk $d \in D^i$ such that $d \cap V_g = \{v\}$. It is not difficult to prove that for every unit disk $d \in D^i$ such that $d \cap V_g \neq \emptyset$, $d \cap V_g$ must also contain an i-extreme point. We assign the following *type* to every $v \in V_g$:

$$T(v) = \{i \mid v \text{ is } i\text{-extreme}\}.$$

Obviously, there are $2^4 = 16$ possible types, as many as the subsets of $\{1, 2, 3, 4\}$.

We now further restrict the lists in $\mathcal{L}'$ of points in V_g to obtain a family $\mathcal{L}'' = \{L''_v\}_{v \in V_g}$ with the following properties:

1. For every $v \in V_g$, $L''_v \subset L'_v$.

2. For every $v \in V_g$, $|L''_v| \geq c'' \ln n$.

3. If $T(v) \neq T(u)$, then $L''_v \cap L''_u = \emptyset$.

The value of the constant c'' will be determined later. Again, the construction of this family $\mathcal{L}''$ is random and is similar to the ones mentioned above.

For any $i \in \{1, \ldots, 4\}$, consider the set $V^i \subset V_g$ of i-extreme points. We remark that V^1, V^2, V^3, V^4 is not necessarily a partition of the extreme points, since, for example, a point may be, say, 1-extreme as well as 2-extreme. We will prove that the following hypergraph is a sub-hypergraph of a discrete interval hypergraph: $H^i = (V^i, \mathcal{E}(H^i))$, where

$$\mathcal{E}(H^i) = \{d \cap V^i \mid d \in D^i,\, d \cap V^i \neq \emptyset\}.$$

Indeed, for every point $v \in V^i$ consider the slope $\theta = \theta_v$ of the line $o^i v$, which is between the slopes of the two halflines bounding the cone K^i. There can be no two different points $v, v' \in V^i$ with $\theta_v = \theta_{v'}$, because if v is, say, inside the segment $o^i v'$, then every disk in D^i that contains v' also contains v, a contradiction to v' being extremal for some disk in D^i. Therefore, vertices in V^i can be put in order of increasing slope; call this order v^i_1, v^i_2, $\ldots$, $v^i_{|V^i|}$. Moreover, the boundary ∂d of a disk $d \in D^i$ in K^i is a θ-monotone curve in the polar coordinates system with center o^i, because d is convex and contains the center of polar coordinates o^i.

We will prove that for $d \in D^i$, $d \cap V^i = \{v^i_x \mid s \leq x \leq t\}$, for some $s, t \in [|V^i|]$ with $s \leq t$. Assume, to the contrary, that there is a disk $d \in D^i$ containing v^i_j and v^i_l, but not v^i_k for $j < k < l$. Then ∂d is above v^i_j and v^i_l and below v^i_k (with respect to o^i). But, since v^i_k is an extreme point for some disk $d' \in D^i$, $\partial d'$ is below v^i_j and v^i_l and above v^i_k. But this implies that ∂d and $\partial d'$ intersect at two points, which is a contradiction.

Since for each $i \in \{1, \ldots, 4\}$, $V^i \subset V_g$, each hypergraph H^i is a subhypergraph of a discrete interval hypergraph on V_g. Therefore, by theorem 3.3, the hypergraph

$$H = (V_g, \mathcal{E}(H^1) \cup \mathcal{E}(H^2) \cup \mathcal{E}(H^3) \cup \mathcal{E}(H^4))$$

can be cf-colored from lists of size $c(4) \ln|V_g|$. If $c'' \geq c(4)$, then $c'' \ln n \geq c(4) \ln|V_g|$ and the hypergraph H admits a cf-coloring from the family $\mathcal{L}''$; call this coloring C.

We will prove that C is also a conflict-free coloring of the points in V_g with respect to unit discs. Indeed, consider any disk d with $d \cap V_g \neq \emptyset$. For some $i \in \{1, 2, 3, 4\}$, we have $d \in D^i$. Consider the set of i-extreme points in d, i.e., $d \cap V^i$, and the set of non-i-extreme points in d. The set of i-extreme points in d is non-empty because $d \in D^i$. Moreover, $d \cap V^i \in \mathcal{E}(H^i)$ and thus there is a vertex $v \in d \cap V^i$ with uniquely occurring color among i-extreme points of d. The color $C(v)$ does not occur in any non-i-extreme point u of d, because $T(v) \neq T(u)$ and thus $L''_v \cap L''_u = \emptyset$. Therefore, the color $C(v)$ occurs uniquely among all points in $d \cap V_g$. This completes the proof of the theorem. □

We next turn our attention to a more difficult task of obtaining good asymptotic upper bounds on the cf-choice number of hypergraphs induced by n points in the plane with respect to arbitrary size discs. Recall that $f = f(n)$ denotes the minimum number such that for any planar set P of n points we have $ch_{\text{cf}}(H(P)) \leq f(n)$.

We obtain the following weak upper bound on $f(n)$.

Theorem 4.4. *$f(n) = O(\sqrt{n})$.*

Later, in Section 5 we show an improved upper bound of $O(\log^2 n)$. Nevertheless, in contrast with our previous proofs that rely on probabilistic arguments, we employ a fully deterministic method to do the coloring. We use the proof technique to show an asymptotically tight upper bound on the cf-choice number of a hypergraph whose vertices are the vertices of some planar graph and whose hyperedges are all subsets of vertices that form a simple path in the graph.

Before proceeding to the proof, we need the following geometric lemma. We feel that this lemma is of independent interest.

Lemma 4.5. *Let P be a set of n points in the plane. Then there exists a partition $P = R \cup B \cup S$ of P into pairwise disjoint sets such that:*

1. $|R| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$, $|S| \leq \sqrt{6}\sqrt{n}$
2. *For every disc d with $d \cap R \neq \emptyset$ and $d \cap B \neq \emptyset$ we also have $d \cap S \neq \emptyset$*

Proof. We assume without loss of generality that the set P is in general position in the sense that no three points are co-linear and no four points are co-circular. If P is not in general position, then we can slightly perturb the points in P and obtain a set P' in general position such that every subset in P that can be cut-off by some disc d corresponds to some subset of P' that can be cut-off by some disc. In other words, in P' the family of "relevant" subsets (i.e., hyperedges realized by discs) contains the family of hyperedges for P. Any partition of P' with the properties required in the lemma also serves as a valid partition of P.

Consider the Delaunay graph $G = G(P)$ on P where two points p and q form an edge in G if and only if there exists a disc d such that $d \cap P = \{p, q\}$. That is, there exists a disc d that cuts off p and q from P. It is a well known fact that G is planar; see, e.g., [15]. Hence, by the Lipton-Tarjan separator theorem [24] and in particular by the version of the separator theorem from [16], there exists a partition $P = R \cup B \cup S$ such that $|R|, |B| \leq \frac{2}{3}n$ and $|S| \leq \sqrt{6n}$ and such that there is no edge connecting a point in R with a point in B. In what follows we refer to the set B as the set of 'blue' points and the set R as the set of 'red' points. We claim that such a partition has the properties required by the lemma. Assume to the contrary that there is a disc d such that $d \cap R \neq \emptyset, d \cap B \neq \emptyset$ and $d \cap S = \emptyset$. We will show that such a disc must contain a 'red-blue' Delaunay edge. That is an edge $pq \in E(G)$ with $p \in R$ and $q \in B$, contradicting the separation property of S. Let c denote the center of d. We shrink d about c until the "first" time we obtain a disc d' with a point from P on its boundary. Denote this point by p. Consider the pencil of discs $\{d_x\}$ with center x lying on the line-segment cp and radius $|xp|$. As x moves from c towards p the discs d_x shrinks and p remains a boundary point of all such discs. See Figure 2 for an illustration. Assume without loss of generality that $p \in R$, namely that p is a red point. As we move x from c towards p, let q be the last blue point we meet on the boundary of some disc d_x. Obviously, the interior of d_x does not contain any blue point but might contain many red points. If d_x contains no points in its interior, then either (a) p and q are the only points on the boundary of d_x, we are done as pq is a red-blue Delaunay edge, a contradiction, or (b) there is a third point r on the boundary of d_x (remember that points are in general position, i.e., no four points are co-circular); but then, we can consider a disc passing through the following three points p, q, and a point r' very close to r on the segment xr, so that this disc contains only p and q from P, which implies the red-blue Delaunay edge pq, a contradiction. Now, suppose the interior of d_x contains some red point. In this case, we look at a pencil of discs $\{d_y\}$ with center y lying on the line segment xq. As we move y from x to q, d_y shrinks but always contains q on its boundary. Thus, there exists a y such that the disc d_y contains some point $p' \in P$ on its boundary and no other points of P in its interior. At this stage the disc d_y contains at least one red point (namely, p') and at least one blue point (namely, q) on its boundary and no points of P in its interior. To finish the argument we need to show that there is a disc d containing only one blue boundary point and one red boundary point of d_y. Assume there is a third point r on the boundary of the disc d_y (there can be no more points, because points are in general position, i.e., no four points are co-circular); then, as before, we can consider a disc passing through p', q and a point r' very close to r on the segment yr, so that this disc contains only p' and q from P. Thus, in any case, there is a red-blue Delaunay edge, a contradiction. This completes the proof of the lemma. □

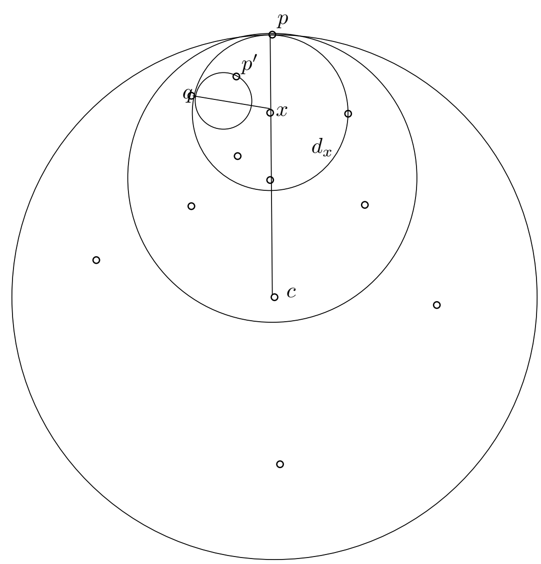

Figure 2: An illustration of the shrinking argument.

Proof of theorem 4.4. The proof uses a 'separator-tree' structure and is algorithmic. That is, given a planar set P of size n together with a family $\mathcal{L}$ of sets of size $c\sqrt{n}$ where c is some absolute constant to be revealed later, we produce a cf-coloring C for P (with respect to discs) with colors from $\mathcal{L}$. That is, $C(p) \in L_p$ where $L_p \in \mathcal{L}$ is the set associated with p.

The algorithm is recursive. We find a partition of $P = R \cup B \cup S$, as in Lemma 4.5. We color all points in S with distinct colors. This can be done greedily as follows: Arbitrarily order the points in S and for each point p along this order choose a color from L_p to assign to p which is distinct from all colors assigned to previous points. This is possible if $|L_p| = c\sqrt{n} \geq \sqrt{6n} \geq |S|$. Next, for each point $q \in R \cup B$ modify the lists $\{L_q\}_{q \in R \cup B}$ by erasing all colors used for S, namely put $\mathcal{L}' = \{\ L_q \setminus \{C(p) \mid p \in S\}\}_{q \in R \cup B}$. We recursively color B and R from $\mathcal{L}'$. Note that the colors assigned to points in $R \cup B$ are distinct from all colors used for S. Note also that if this coloring is indeed a valid cf-coloring of P from $\mathcal{L}$ then the function $f(n)$ which is the minimum cf-choice number for n points in the plane satisfies the following recursive inequality:

$$f(n) \leq \sqrt{6}\sqrt{n} + f(2n/3) \leq \sum_{i=0}^{\infty} \sqrt{6 \cdot \left(\frac{2}{3}\right)^i n} = \frac{\sqrt{6}\sqrt{n}}{1 - \sqrt{2/3}} \approx 13.3485\sqrt{n}$$

Thus, we have $f(n) \leq c\sqrt{n}$, for $c \approx 13.3485$, as claimed. □

In the next section, we improve on the $O(\sqrt{n})$ bound of theorem 4.4, albeit using a probabilistic approach.

Given a simple graph $G = (V, E)$, consider the hypergraph

$$H_G = (V, \{S \mid S \text{ is the vertex set of a simple path in } G\}).$$

The proof of the following theorem is similar to the proof of theorem 4.4.

Theorem 4.6. *Let G be a planar graph with n vertices. Then $ch_{cf}(H_G) = O(\sqrt{n})$.*

Remark 4.7. The upper bound $O(\sqrt{n})$ is asymptotically tight, since for the $\sqrt{n}\times\sqrt{n}$ grid graph $G_{\sqrt{n}}$, it was proved in [11] that $\chi_{\mathrm{cf}}(H_{G_{\sqrt{n}}}) = \Omega(\sqrt{n})$ and thus, from inequality (2), also $ch_{\mathrm{cf}}(H_{G_{\sqrt{n}}}) = \Omega(\sqrt{n})$.

5 A connection between choosability and colorability in general hypergraphs

In this section we prove the following theorem.

Theorem 5.1. *For every hypergraph H, $ch_{cf}(H) \leq \chi_{cf}(H) \cdot \ln n + 1$.*

Proof. If $k = \chi_{\mathrm{cf}}(H)$, there is a cf-coloring C of H with colors $\{1, \dots, k\}$, which induces a partition of V into k classes: $V_1 \cup V_2 \cup \cdots \cup V_k$. Consider a family $\mathcal{L} = \{L_v\}_{v\in V}$, such that for every v, $|L_v| = k^* > k \cdot \ln n$. We wish to find a family $\mathcal{L}' = \{L'_v\}_{v\in V}$ with the following properties:

1. For every $v \in V$, $L'_v \subset L_v$.
2. For every $v \in V$, $L'_v \neq \emptyset$.
3. For every $i \neq j$, if $v \in V_i$ and $u \in V_j$, then $L'_v \cap L'_u = \emptyset$.

Obviously, if such a family $\mathcal{L}'$ exists, then there exists a cf-coloring from $\mathcal{L}'$: For each $v \in V$, pick a color $x \in L'_v$ and assign it to v.

Once again, the family $\mathcal{L}'$ is created randomly as follows: For each element in $\cup\mathcal{L}$, assign it uniformly at random to one of the k classes of the partition $V_1 \cup \cdots \cup V_k$. For every vertex $v \in V$, say with $v \in V_i$, we create L'_v, by keeping only elements of L_v that were assigned through the above random process to v's class, V_i.

The family $\mathcal{L}'$ has properties 1 and 3. We will prove that with positive probability it also has property 2.

For a fixed v, the probability that $L'_v = \emptyset$ is at most

$$\left(1 - \frac{1}{k}\right)^{k^*} \leq e^{-k^*/k} < e^{-\ln n} = \frac{1}{n}$$

and therefore, using the union bound, the probability that for at least one vertex v, $L'_v = \emptyset$, is at most

$$n\left(1 - \frac{1}{k}\right)^{k^*} < 1.$$

Thus, there is at least one family $\mathcal{L}'$ where property 2 also holds, as claimed. □

Corollary 5.2. *For a hypergraph H of n points with respect to arbitrary discs, $ch_{cf}(H) = O(\log^2 n)$.*

Proof. Immediate from the fact that $\chi_{\text{cf}}(H) = O(\log n)$ (see [18]) and Theorem 5.1. $\square$

Remark 5.3. Observe that in the proof of theorem 5.1 we used the property of cf-coloring in a very "weak" sense. Our proof in fact says that any 'good' (non-list) coloring induces a 'good' list-coloring (with logarithmic times more colors). The argument in the proof of theorem 5.1 is a generalization of an argument first given in [17], to prove that any bipartite graph with n vertices is $O(\log n)$-choosable (see also [2]). In particular, we also have:

Theorem 5.4. *For every hypergraph H, $ch(H) \leq \chi(H) \cdot \ln n + 1$.*

6 Open problems

We consider the following as interesting problems left open here:

- Let H be a hypergraph induced by n points in the plane with respect to discs. Close the gap between the lower bound of $\Omega(\log n)$ and the upper bound of $O(\log^2 n)$ on $ch_{\text{cf}}(H)$.
- Let H be a hypergraph induced by n axis-parallel rectangles in the plane. Is it true that $ch(H) = O(\log n)$? It is known that $\chi(H) = \Theta(\log n)$ [25, 29].

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